

## ON THE CLASS OF CAUSTICS BY REFLECTION

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ABSTRACT. Given any light position  $\mathcal{S} \in \mathbb{P}^2$  and any algebraic curve  $\mathcal{C}$  of  $\mathbb{P}^2$  (with any kind of singularities), we consider the incident lines coming from  $\mathcal{S}$  (i.e. the lines containing  $\mathcal{S}$ ) and their reflected lines after reflection off the mirror curve  $\mathcal{C}$ . The caustic by reflection  $\Sigma_{\mathcal{S}}(\mathcal{C})$  is the Zariski closure of the envelope of these reflected lines. We introduce the notion of reflected polar curve and express the class of  $\Sigma_{\mathcal{S}}(\mathcal{C})$  in terms of intersection numbers of  $\mathcal{C}$  with the reflected polar curve, thanks to a fundamental lemma established in [14]. This approach enables us to get an explicit formula for the class of  $\Sigma_{\mathcal{S}}(\mathcal{C})$  in every case in terms of intersection numbers of the initial curve  $\mathcal{C}$ .

## INTRODUCTION

In the study of caustics by reflection the cyclic points  $\mathcal{I} := [1 : i : 0]$  and  $\mathcal{J} := [1 : -i : 0]$  play an important role (let us write  $I := (1, i, 0)$  and  $J := (1, -i, 0)$ ). As usual, we denote by  $\ell_{\infty}$  the infinite line of  $\mathbb{P}^2 := \mathbb{P}^2(\mathbb{C})$ .

We consider a light point  $\mathcal{S} = [x_0 : y_0 : z_0] \in \mathbb{P}^2 \setminus \{\mathcal{I}, \mathcal{J}\}$  and  $S := (x_0, y_0, z_0)$ . We consider a mirror given by an irreducible algebraic curve  $\mathcal{C} = V(F)$  of  $\mathbb{P}^2$ , with  $F \in \mathbb{C}[x, y, z]$  a homogeneous polynomial of degree  $d \geq 2$ . We denote by  $d^{\vee}$  the class of  $\mathcal{C}$ . We write  $\Sigma_{\mathcal{S}}(\mathcal{C})$  the caustic by reflection of the mirror curve  $\mathcal{C}$  with source point  $\mathcal{S}$ , i.e. the Zariski closure of the envelope of the reflected lines of incident lines coming from  $\mathcal{S}$  after reflection off  $\mathcal{C}$ . Recall that, when  $\mathcal{S}$  is finite, Quetelet and Dandelin [15, 8] proved that the caustic by reflection  $\Sigma_{\mathcal{S}}(\mathcal{C})$  is the evolute of the  $\mathcal{S}$ -centered homothety (with ratio 2) of the pedal of  $\mathcal{C}$  from  $\mathcal{S}$  (i.e. the evolute of the orthotomic of  $\mathcal{C}$  with respect to  $\mathcal{S}$ ). This decomposition has also been used in a modern approach by [2, 3, 4] to study the source genericity (in the real case).

In [6], Chasles proved that the class of  $\Sigma_{\mathcal{S}}(\mathcal{C})$  is equal to  $2d^{\vee} + d$  for a generic  $(\mathcal{C}, \mathcal{S})$ . In [1], Brocard and Lemoyne gave (without any proof) a more general formula only when  $\mathcal{S} \notin \ell_{\infty}$ . The Brocard and Lemoyne formula appears to be the direct composition of formulas got by Salmon and Cayley in [16, p. 137, 154] for some geometric characteristics of evolute and pedal curves. The formula given by Brocard and Lemoyne is not satisfactory for the following reason. The results of Salmon and Cayley apply only to curves having no singularities other than ordinary nodes and cusps [16, p. 82], but the pedal of such a curve is not necessarily a curve satisfying the same properties. For example, the pedal curve of the rational cubic  $V(y^2z - x^3)$  from  $[4 : 0 : 1]$  is a quartic curve with a triple ordinary point. Therefore it is not correct to compose directly the formulas got by Salmon and Cayley as Brocard and Lemoyne apparently did (see also appendix A for a counterexample of the Brocard and Lemoyne formula for the class of the caustic by reflection).

Let us mention more recent works on the evolute and on its generalization in higher dimension [9, 17, 5]. In [9], Fantechi gave a necessary and sufficient condition for the birationality of the

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evolute of a curve and studies the number and type of the singularities of the general evolute. Let us insist on the fact that there exist irreducible algebraic curves (other than lines and circles) on which the evolute is not birational. This study of evolute is generalized in higher dimension by Trifogli in [17]. Catanese and Trifogli gave general formulas for degrees and classes (with multiplicity) of focal loci of smooth algebraic varieties [5].

The aim of the present paper is to give a general formula for the class (with multiplicity) of the caustic by reflection for any algebraic curve  $\mathcal{C}$  (without any restriction neither on the singularity points nor on the flex points) and for any light point  $\mathcal{S}$  (including the case when  $\mathcal{S}$  is on the infinite line  $\ell_\infty$ ).

In Section 1, we state our general formula for the class of caustics by reflection and illustrate it with the computation of the class of the caustic by reflection of two particular curves (the lemniscate of Bernoulli and the quintic considered in [14]).

In Section 2, we present our approach based on a fundamental lemma for rational map and express the class of caustics in terms of intersection numbers.

In Section 3, we prove our main theorem.

In section 4, we characterize the cases in which  $\Sigma_{\mathcal{S}}(\mathcal{C})$  is reduced to a point.

In appendix A, we compare our formula with the one stated by Brocard and Lemoyne.

In appendix B, we prove a useful formula expressing the classical intersection number in terms of probranches.

## 1. RESULTS ON THE COMPUTATION OF THE CLASS OF CAUSTICS BY REFLECTION

In this paper, we do not consider the cases in which the caustic by reflection  $\Sigma_{\mathcal{S}}(\mathcal{C})$  is a single point. We recall that these cases are easily characterized as follows.

**Proposition 1.** *Assume that*

- (i)  $\mathcal{S} \notin \{\mathcal{I}, \mathcal{J}\}$ ,
- (ii)  $\mathcal{C}$  is not a line (i.e.  $d \neq 1$ ),
- (iii) if  $d = 2$ , then  $\mathcal{S}$  is not a focus of the conic  $\mathcal{C}$ .

*Then  $\Sigma_{\mathcal{S}}(\mathcal{C})$  is not reduced to a point and is an irreducible curve.*

We will define a rational map  $R_{F,\mathcal{S}} : \mathbb{P}^2 \mapsto \mathbb{P}^2$  such that, for a general point  $m \in \mathcal{C}$ ,  $R_{F,\mathcal{S}}(m) \in \mathbb{P}^2$  corresponds to the coefficient of the equation of the reflected line  $\mathcal{R}_m$  (got by reflection of  $(\mathcal{S}m)$  on  $\mathcal{C}$  at  $m$ ). The map  $R_{F,\mathcal{S}}$  may be non birational. For this reason, we introduce the notion of class with multiplicity of  $\Sigma_{\mathcal{S}}(\mathcal{C})$ :

$$\text{mclass}(\Sigma_{\mathcal{S}}(\mathcal{C})) = \delta_1(\mathcal{S}, \mathcal{C}) \times \text{class}(\Sigma_{\mathcal{S}}(\mathcal{C}))$$

where  $\text{class}(\Sigma_{\mathcal{S}}(\mathcal{C}))$  is the degree of the algebraic curve  $\Sigma_{\mathcal{S}}(\mathcal{C})$  and where  $\delta_1(\mathcal{S}, \mathcal{C})$  is the degree of the rational map  $R_{F,\mathcal{S}}$ . We recall that  $\delta_1(\mathcal{S}, \mathcal{C})$  corresponds to the number of preimages on  $\mathcal{C}$  of a generic point of  $\Sigma_{\mathcal{S}}(\mathcal{C})$  by  $R_{F,\mathcal{S}}$ .

Before stating our main result, let us introduce some notations. For every  $m_1 \in \mathbb{P}^2$ , we write  $\mu_{m_1} = \mu_{m_1}(\mathcal{C})$  the multiplicity of  $m_1$  on  $\mathcal{C}$  and consider the set  $\text{Branch}_{m_1}(\mathcal{C})$  of branches of  $\mathcal{C}$  at  $m_1$ . We denote by  $\mathcal{E}$  the set of couples point-branch  $(m_1, \mathcal{B})$  of  $\mathcal{C}$  with  $m_1 \in \mathcal{C}$  and  $\mathcal{B} \in \text{Branch}_{m_1}(\mathcal{C})$ . For every  $(m_1, \mathcal{B}) \in \mathcal{E}$ , we write  $e_{\mathcal{B}}$  the multiplicity of  $\mathcal{B}$  and  $\mathcal{T}_{m_1}(\mathcal{B})$  the tangent line to  $\mathcal{B}$  at  $m_1$ ; we observe that  $\mu_{m_1} = \sum_{\mathcal{B} \in \text{Branch}_{m_1}(\mathcal{C})} e_{\mathcal{B}}$ . We write  $i_{m_1}(\Gamma, \Gamma')$  the

intersection number of two curves  $\Gamma$  and  $\Gamma'$  at  $m_1$ . For any algebraic curve  $\mathcal{C}'$  of  $\mathcal{P}^2$ , we also define the contact number  $\Omega_{m_1}(\mathcal{C}, \mathcal{C}')$  of  $\mathcal{C}$  and  $\mathcal{C}'$  at  $m_1 \in \mathbb{P}^2$  by

$$\Omega_{m_1}(\mathcal{C}, \mathcal{C}') := i_{m_1}(\mathcal{C}, \mathcal{C}') - \mu_{m_1}(\mathcal{C})\mu_{m_1}(\mathcal{C}') \quad \text{if } m_1 \in \mathcal{C} \cap \mathcal{C}'$$

and

$$\Omega_{m_1}(\mathcal{C}, \mathcal{C}') := 0 \quad \text{if } m_1 \notin \mathcal{C} \cap \mathcal{C}'.$$

Recall that  $\Omega_{m_1}(\mathcal{C}, \mathcal{C}') = 0$  means that  $m_1 \notin \mathcal{C} \cap \mathcal{C}'$  or that  $\mathcal{C}$  and  $\mathcal{C}'$  intersect transversally at  $m_1$ .

**Theorem 2.** *Assume that the hypotheses of Proposition 1 hold true.*

(1) *If  $\mathcal{S} \notin \ell_\infty$ , the class (with multiplicity) of  $\Sigma_{\mathcal{S}}(\mathcal{C})$  is given by*

$$mclass(\Sigma_{\mathcal{S}}(\mathcal{C})) = 2d^\vee + d - 2f' - g - f - g' - q', \quad (1)$$

where

- $g$  is the contact number of  $\mathcal{C}$  with  $\ell_\infty$ , i.e.

$$g := \sum_{m_1 \in \mathcal{C} \cap \ell_\infty} \Omega_{m_1}(\mathcal{C}, \ell_\infty),$$

- $f$  is the multiplicity number at a cyclic point of  $\mathcal{C}$  with an isotropic line of  $\mathcal{S}$ , i.e.

$$f := i_{\mathcal{I}}(\mathcal{C}, (\mathcal{I}\mathcal{S})) + i_{\mathcal{J}}(\mathcal{C}, (\mathcal{J}\mathcal{S})),$$

- $f'$  is the contact number of  $\mathcal{C}$  with an isotropic line outside  $\{\mathcal{I}, \mathcal{J}, \mathcal{S}\}$ , i.e.

$$f' := \sum_{m_1 \in (\mathcal{C} \cap (\mathcal{I}\mathcal{S})) \setminus \{\mathcal{I}, \mathcal{S}\}} \Omega_{m_1}(\mathcal{C}, (\mathcal{I}\mathcal{S})) + \sum_{m_1 \in (\mathcal{C} \cap (\mathcal{J}\mathcal{S})) \setminus \{\mathcal{J}, \mathcal{S}\}} \Omega_{m_1}(\mathcal{C}, (\mathcal{J}\mathcal{S})),$$

- $g'$  given by

$$g' := i_{\mathcal{S}}(\mathcal{C}, (\mathcal{I}\mathcal{S})) + i_{\mathcal{S}}(\mathcal{C}, (\mathcal{J}\mathcal{S})) - \mu_{\mathcal{S}};$$

- $q'$  is given by

$$q' := \sum_{(m_1, \mathcal{B}) \in \mathcal{E}: m_1 \notin \{\mathcal{I}, \mathcal{J}, \mathcal{S}\}, T_{m_1} \mathcal{B} = (\mathcal{I}\mathcal{S}) \text{ or } T_{m_1} \mathcal{B} = (\mathcal{J}\mathcal{S}), i_{m_1}(\mathcal{B}, T_{m_1}(\mathcal{B})) \geq 2e_{\mathcal{B}}} [i_{m_1}(\mathcal{B}, T_{m_1}(\mathcal{B})) - 2e_{\mathcal{B}}].$$

(2) *If  $\mathcal{S} \in \ell_\infty$ , the class of  $\Sigma_{\mathcal{S}}(\mathcal{C})$  is*

$$mclass(\Sigma_{\mathcal{S}}(\mathcal{C})) = 2d^\vee + d - 2g - \mu_{\mathcal{I}} - \mu_{\mathcal{J}} - \mu_{\mathcal{S}} - c'(\mathcal{S}), \quad (2)$$

with

$$c'(\mathcal{S}) := \sum_{\mathcal{B} \in \text{Branch}_{\mathcal{S}}(\mathcal{C}): i_{\mathcal{S}}(\mathcal{B}, \ell_\infty) = 2e_{\mathcal{B}}} (e_{\mathcal{B}} + \min(i_{\mathcal{S}}(\mathcal{B}, \text{Osc}_{\mathcal{S}}(\mathcal{B})) - 3e_{\mathcal{B}}, 0)),$$

where  $\text{Osc}_{\mathcal{S}}(\mathcal{B})$  is any smooth algebraic osculating curve to  $\mathcal{B}$  at  $\mathcal{S}$  (i.e. any smooth algebraic curve  $\mathcal{C}'$  such that  $i_{\mathcal{S}}(\mathcal{B}, \mathcal{C}') > 2e_{\mathcal{B}}$ ).

The notations introduced in this theorem are directly inspired by those of Salmon and Cayley [16]. Let us point out that, in this article,  $g$  is not the geometric genus of the curve.

Observe that, when  $i_{\mathcal{S}}(\mathcal{B}, T_{\mathcal{S}}(\mathcal{B})) = 2e_{\mathcal{B}}$ , we have  $\min(i_{\mathcal{S}}(\mathcal{B}, \text{Osc}_{\mathcal{S}}(\mathcal{B})) - 3e_{\mathcal{B}}, 0) = 0$  except if  $\mathcal{S}$  is a singular point and if the probranches of  $\mathcal{B}$  are given by  $Y - x_0^{-1}y_0 = \alpha Z^2 + \alpha_1 Z^{\beta_1} + \dots$  in the chart  $X = 1$  if  $x_0 \neq 0$  (or  $X - y_0^{-1}x_0 = \alpha Z^2 + \alpha_1 Z^{\beta_1} + \dots$  in the chart  $Y = 1$  otherwise), with  $\alpha \neq 0$ ,  $\alpha_1 \neq 0$  and  $2 < \beta_1 < 3$ . Hence  $c'(\mathcal{S}) = \sum_{\mathcal{B} \in \text{Branch}_{\mathcal{S}}(\mathcal{C}): i_{\mathcal{S}}(\mathcal{B}, \ell_\infty) = 2e_{\mathcal{B}}} e_{\mathcal{B}}$  when  $\mathcal{C}$  admits no such branch tangent at  $\mathcal{S}$  to  $\ell_\infty$ .

**Remark 3.** For a generic irreducible plane curve  $\mathcal{C}$  of degree  $d$  and for a generic source point  $\mathcal{S}$ , we have  $\delta_1(\mathcal{S}, \mathcal{C}) = 1$  and so  $\text{mclass}(\Sigma_{\mathcal{S}}(\mathcal{C})) = \text{class}(\Sigma_{\mathcal{S}}(\mathcal{C}))$ .

Indeed, given a curve  $\mathcal{C}$ , for  $\mathcal{S}$  generic (outside  $\ell_{\infty}$ ), according to Quetelet and Dandelin [15, 8], the caustic by reflection  $\Sigma_{\mathcal{S}}(\mathcal{C})$  is the evolute of the  $\mathcal{S}$ -centered homothety (with ratio 2) of the pedal of  $\mathcal{C}$  from  $\mathcal{S}$  (i.e. the evolute of the orthotomic of  $\mathcal{C}$  with respect to  $\mathcal{S}$ ).

Now, for  $\mathcal{C}$  generic (when  $\mathcal{C}$  is not a line), the natural map from  $\mathcal{C}$  to its pedal  $\mathcal{P}$  is birational (since it admits a finite number of bitangents). Moreover, the  $\mathcal{S}$ -centered homothety with ratio 2 is an isomorphism and, according to [9], the natural map from an algebraic curve to its evolute is generically birational (and that, when it is not birational, it is generically  $2:1$ ).

To conclude, we recall that an algebraic irreducible curve of  $\mathbb{P}^2$  is generically the pedal of an algebraic curve (it is the pedal curve of its negative pedal also called orthocaustic).

Let us now apply our result by computing the class of caustics by reflection for two particular mirror curves.

**Example of the lemniscate of Bernoulli.** We consider the case when  $\mathcal{C} = V(F)$  is the lemniscate of Bernoulli given by  $F(x, y, z) = (x^2 + y^2)^2 - 2(x^2 - y^2)z^2$  and  $\mathcal{S} \in \mathbb{P}^2 \setminus \{\mathcal{I}, \mathcal{J}\}$ . The degree of  $\mathcal{C}$  is  $d = 4$ . The singular points of  $\mathcal{C}$  are :  $\mathcal{I} = [1 : i : 0]$ ,  $\mathcal{J} = [1 : -i : 0]$  and  $\mathcal{O} = [0 : 0 : 1]$ . These three points are double points, each one having two different tangent lines. Hence the class of  $\mathcal{C}$  is given by  $d^{\vee} = d(d - 1) - 3 \times 2 = 6$  and so

$$2d^{\vee} + d = 16.$$

The tangent lines to  $\mathcal{C}$  at  $\mathcal{I}$  are  $\ell_{1,\mathcal{I}} := V(Y - iZ - iX)$  and  $\ell_{2,\mathcal{I}} := V(Y - iZ + iX)$  (the intersection number of  $\mathcal{C}$  with  $\ell_{1,\mathcal{I}}$  or with  $\ell_{2,\mathcal{I}}$  at  $\mathcal{I}$  is equal to 4). The tangent lines to  $\mathcal{C}$  at  $\mathcal{J}$  are  $\ell_{1,\mathcal{J}} := V(Y + iZ - iX)$  and  $\ell_{2,\mathcal{J}} := V(Y + iZ + iX)$  (the intersection number of  $\mathcal{C}$  with  $\ell_{1,\mathcal{J}}$  or with  $\ell_{2,\mathcal{J}}$  at  $\mathcal{J}$  is equal to 4). This ensures that we have

$$f = 2(2 + \mathbf{1}_{\mathcal{S} \in \ell_{1,\mathcal{I}}} + \mathbf{1}_{\mathcal{S} \in \ell_{2,\mathcal{I}}} + \mathbf{1}_{\mathcal{S} \in \ell_{1,\mathcal{J}}} + \mathbf{1}_{\mathcal{S} \in \ell_{2,\mathcal{J}}}).$$

Observe that  $\ell_{\infty}$  is not tangent to  $\mathcal{C}$ . Indeed  $\mathcal{I}$  and  $\mathcal{J}$  are the only points in  $\mathcal{C} \cap \ell_{\infty}$  and  $\ell_{\infty}$  is not tangent to  $\mathcal{C}$  at these points. Therefore we have  $g = 0$  and  $c'(\mathcal{S}) = 0$ .

Since  $\mathcal{I}$  and  $\mathcal{J}$  are also the only points at which  $\mathcal{C}$  is tangent to an isotropic line (i.e. a line containing  $\mathcal{I}$  or  $\mathcal{J}$ ), we have  $f' = 0$ ,  $g' = \mu_{\mathcal{S}}$ ,  $q' = 0$ . In this case, one can check that  $\delta_1(\mathcal{S}, \mathcal{C}) = 1$ . Finally, we get

$$\text{if } \mathcal{S} \notin \ell_{\infty}, \quad \text{class}(\Sigma_{\mathcal{S}}(\mathcal{C})) = 12 - 2(\mathbf{1}_{\mathcal{S} \in \ell_{1,\mathcal{I}} \cup \ell_{2,\mathcal{I}}} + \mathbf{1}_{\mathcal{S} \in \ell_{1,\mathcal{J}} \cup \ell_{2,\mathcal{J}}}) - \mu_{\mathcal{S}}. \quad (3)$$

Moreover, since  $\mu_{\mathcal{I}} = \mu_{\mathcal{J}} = 2$ , we have

$$\text{if } \mathcal{S} \in \ell_{\infty} \setminus \{\mathcal{I}, \mathcal{J}\}, \quad \text{class}(\Sigma_{\mathcal{S}}(\mathcal{C})) = 16 - 2 - 2 = 12, \quad (4)$$

(since  $\mu_{\mathcal{I}} = \mu_{\mathcal{J}} = 2$  and since  $\mu_{\mathcal{S}} = 0$ ) For example, when  $\mathcal{S} = [1 : 0 : 1]$ , we get  $\text{class}(\Sigma_{\mathcal{S}}(\mathcal{C})) = 8$ , since  $\mathcal{S}$  is in  $\ell_{2,\mathcal{I}} \cap \ell_{1,\mathcal{J}}$  but not in  $\mathcal{C}$  (so  $\mu_{\mathcal{S}} = 0$ ).

**1.1. Another example.** As in [14], we consider the quintic curve  $\mathcal{C} = V(F)$  with  $F(x, y, z) = y^2 z^3 - x^5$ . We also consider a light point  $\mathcal{S} = [x_0 : y_0 : z_0] \in \mathbb{P}^2 \setminus \{\mathcal{I}, \mathcal{J}\}$ . This curve admits two singular points:  $A_1 := [0 : 0 : 1]$  and  $A_2 := [0 : 1 : 0]$ , we have  $d = 5$ .

We recall that,  $\mathcal{C}$  admits a single branch at  $A_1$ , which has multiplicity 2 and which is tangent to  $Y = 0$ . We observe that  $i_{A_1}(\mathcal{C}, V(Y)) = 5$ .

Analogously,  $\mathcal{C}$  admits a single branch at  $A_2$ , which has multiplicity 3 and which is tangent to  $\ell_{\infty}$ . We observe that  $i_{A_2}(\mathcal{C}, \ell_{\infty}) = 5$ .

We get that the class of  $\mathcal{C}$  is  $d^\vee = 5$  and that  $\mathcal{C}$  has no inflexion point (these two facts are proved in [14]). In particular, we get that  $2d^\vee + d = 15$ .

Since  $A_2$  is the only point of  $\mathcal{C} \cap \ell_\infty$ , we get that  $g = \Omega_{A_2}(\mathcal{C}, \ell_\infty) = 2$  and  $f = 0$ .

The curve  $\mathcal{C}$  admits six (pairwise distinct) isotropic tangent lines other than  $\ell_\infty$ :  $\ell_1, \ell_2$  and  $\ell_3$  containing  $\mathcal{I}$

$$\forall k \in \{1, 2, 3\}, \quad \ell_k = V\left(iX - Y + \frac{3i}{25}\alpha^k \sqrt[3]{20Z}\right), \quad \text{with } \alpha := e^{\frac{2i\pi}{3}}$$

and  $\ell_4, \ell_5$  and  $\ell_6$  containing  $\mathcal{J}$ :

$$\forall k \in \{1, 2, 3\}, \quad \ell_{3+k} = V\left(iX + Y + \frac{3i}{25}\alpha^k \sqrt[3]{20Z}\right).$$

For every  $i \in \{1, 2, 3, 4, 5, 6\}$ , we write  $a_i$  the point at which  $\mathcal{C}$  is tangent to  $\ell_i$  (the points  $a_i$  correspond to the points of  $\mathcal{C} \cap V(F_x^2 + F_y^2) \setminus \{A_1, A_2\}$ ). Since  $\mathcal{C}$  contains no inflexion point and since  $A_1$  and  $A_2$  are the only singular points of  $\mathcal{C}$ , we get that,

$$f' = \#\{i \in \{1, 2, 3, 4, 5, 6\} : \mathcal{S} \in \ell_i \setminus \{a_i\}\} \quad \text{and} \quad q' = 0$$

when  $\mathcal{S} \notin \ell_\infty$ .

Now recall that  $g' = i_{\mathcal{S}}(\mathcal{C}, (\mathcal{I}\mathcal{S})) + i_{\mathcal{S}}(\mathcal{C}, (\mathcal{J}\mathcal{S})) - \mu_{\mathcal{S}}$ . Again, in this case, one can check that  $\delta_1(\mathcal{S}, \mathcal{C}) = 1$ . If  $\mathcal{S} \notin \ell_\infty$ , we have

$$\text{class}(\Sigma_{\mathcal{S}}(\mathcal{C})) = 13 - 2 \times \#\{i \in \{1, 2, 3, 4, 5, 6\} : \mathcal{S} \in \ell_i \setminus \{a_i\}\} - g' \quad (5)$$

and if  $\mathcal{S} \in \ell_\infty \setminus \{\mathcal{I}, \mathcal{J}\}$ , we have

$$\text{class}(\Sigma_{\mathcal{S}}(\mathcal{C})) = 11 - 3 \times \mathbf{1}_{\mathcal{S}=A_2}. \quad (6)$$

We observe that the points of  $\mathbb{P}^2 \setminus \{\mathcal{I}, \mathcal{J}\}$  belonging to two distinct  $\ell_k$  are outside  $\mathcal{C}$ . The set of these points is

$$\mathcal{E} := \bigcup_{k=1}^3 \left\{ \left[ -\frac{3}{25} \sqrt[3]{20}\alpha^k : 0 : 1 \right], \left[ \frac{3}{50} \sqrt[3]{20}\alpha^k : \frac{3}{50} \sqrt{3} \sqrt[3]{20}\alpha^k : 1 \right], \left[ \frac{3}{50} \sqrt[3]{20}\alpha^k : -\frac{3}{50} \sqrt{3} \sqrt[3]{20}\alpha^k : 1 \right] \right\}$$

with  $\alpha = e^{\frac{2i\pi}{3}}$ . Finally, the class of the caustic in the different cases is summarized in the following table.

Condition on $\mathcal{S} \in \mathbb{P}^2 \setminus \{\mathcal{I}, \mathcal{J}\}$	$\text{class}(\Sigma_{\mathcal{S}}(\mathcal{C})) =$
$\mathcal{S} = A_2$	8
$\mathcal{S} \in \mathcal{E}$	9
$\mathcal{S} \in \mathcal{C} \cap \bigcup_{k=1}^6 (\ell_k \setminus \{a_k\})$	10
$\mathcal{S} \in (\ell_\infty \setminus \{A_2\}) \cup \left( \bigcup_{k=1}^6 \ell_k \setminus (\mathcal{E} \cup \mathcal{C}) \right) \cup \{A_1\} \cup \{a_1, \dots, a_6\}$	11
$\mathcal{S} \in \mathcal{C} \setminus \left( \ell_\infty \cup \{A_1\} \cup \bigcup_{k=1}^6 \ell_k \right)$	12
otherwise	13

## 2. CLASS OF CAUSTICS EXPRESSED IN TERMS OF INTERSECTION NUMBERS

Assume that the hypotheses of Proposition 1 hold true.

We write  $\text{Sing}(\mathcal{C})$  the set of singular points of  $\mathcal{C}$  and  $\text{Reg}(\mathcal{C})$  the set of nonsingular points of  $\mathcal{C}$ . Let us recall that we proved in [14] that, for every  $m = [x : y : z] \in \text{Reg}(\mathcal{C}) \setminus (\{\mathcal{S}\} \cup$

$\ell_\infty$ ), the reflected line  $\mathcal{R}_m$  of the incident line  $(\mathcal{S}m)$  after reflection off  $\mathcal{C}$  is given by  $\mathcal{R}_m = V(\langle \tilde{\rho}_{F,S}(x, y, z), \cdot \rangle)^1$  with

$$\tilde{\rho}_{F,S}(x, y, z) = \begin{pmatrix} z(z_0y - zy_0)(F_x^2 - F_y^2) + 2z(zx_0 - z_0x)F_xF_y \\ z(z_0x - zx_0)(F_x^2 - F_y^2) + 2z(z_0y - zy_0)F_xF_y \\ (xyz_0 + yzx_0 - 2z_0xy)(F_x^2 - F_y^2) + 2(yzy_0 - xzx_0 + z_0x^2 - z_0y^2)F_xF_y \end{pmatrix}.$$

Now using the fact that  $xF_x + yF_y + zF_z = 0$  on  $\mathcal{C}$ , we get that

$$\tilde{\rho}_{F,S}(x, y, z) = z \cdot \tilde{R}_{F,S}(x, y, z),$$

with

$$\tilde{R}_{F,S}(x, y, z) := \begin{pmatrix} (z_0y - zy_0)(F_x^2 + F_y^2) + 2z\Delta_SF.F_y \\ (zx_0 - z_0x)(F_x^2 + F_y^2) - 2z\Delta_SF.F_x \\ (xy_0 - yx_0)(F_x^2 + F_y^2) - 2\Delta_SF(-xF_y - yF_x) \end{pmatrix},$$

with  $\Delta_{(x_1, y_1, z_1)}F := x_1F_x + y_1F_y + z_1F_z$ . This can be rewritten <sup>2</sup>

$$\tilde{R}_{F,S} = id \wedge [\Delta_IF\Delta_JF.S - \Delta_SF\Delta_IF.J - \Delta_SF\Delta_JF.I].$$

We observe that  $\tilde{R}_{F,S}$  defines a rational map  $R_{F,S}$  and that its coordinates  $R_{F,S,1}, R_{F,S,2}, R_{F,S,3}$  are homogeneous polynomials of  $\mathbb{C}[x, y, z]$  of degree  $2d - 1$ . As usual we write  $Base((R_{F,S})|_{\mathcal{C}}) := \mathcal{C} \cap V(R_{F,S,1}) \cap V(R_{F,S,2}) \cap V(R_{F,S,3})$ . For completeness, we mention that we can easily identify these base points (by solving a simple linear system).

**Proposition 4.** *The base points of  $(R_{F,S})|_{\mathcal{C}}$  are the following:*

$\mathcal{I}, \mathcal{J}, \mathcal{S}$  (if these points are in  $\mathcal{C}$ ), the singular points of  $\mathcal{C}$  and the non-singular points of  $\mathcal{C}$  with tangent line  $(\mathcal{IS}), (\mathcal{JS})$  or  $(\mathcal{IJ})$  ( $= \ell_\infty$ ).

Observe that this result is also a direct consequence of our technical Lemma 9. Indeed these base points are the points  $m_1 \in \mathcal{C}$  such that  $V_{m_1} + h_{m_1} \neq 0$  (see section 3).

In [14], we have defined a rational map  $\Phi_{F,S}$  mapping every  $m \in \mathcal{C}$  to the corresponding point of  $\Sigma_S(\mathcal{C})$  and such that  $\Sigma_S(\mathcal{C})$  is the Zariski closure of  $\Phi_{F,S}(\mathcal{C})$ .

**Proposition 5.** *We have*

$$class(\Sigma_S(\mathcal{C})) = deg(\overline{R_{F,S}(\mathcal{C})}), \quad (7)$$

where  $\overline{R_{F,S}(\mathcal{C})}$  stands for the Zariski closure of  $R_{F,S}(\mathcal{C})$ .

*Proof.* This comes from the fact that  $\Sigma_S(\mathcal{C})$  is the Zariski closure of the envelope of  $\{\mathcal{R}_m, m \in Reg(\mathcal{C}) \setminus (\{\mathcal{S}\} \cup \ell_\infty)\}$  and can be precised as follows. Let us write, for every algebraic curve  $\Gamma = V(G)$  (with homogeneous  $G$  in  $\mathbb{C}[x, y, z]$ ), the Gauss map  $\delta_\Gamma : \mathbb{P}^2 \rightarrow \mathbb{P}^2$  defined by  $\delta_\Gamma([x : y : z]) = [G_x : G_y : G_z]$ , we get immediately the diagram

$$\begin{array}{ccc} \mathbb{P}^2 & \xrightarrow{\Phi_{F,S}} & \mathbb{P}^2 \\ & \searrow R_{F,S} & \downarrow \delta_{\Sigma_S(\mathcal{C})} \\ & & \mathbb{P}^2 \end{array}$$

<sup>1</sup>with  $\langle (a, b, c), (X, Y, Z) \rangle = aX + bY + cZ$

<sup>2</sup>with notation  $(x_1, y_1, z_1) \wedge (x_2, y_2, z_2) = \begin{pmatrix} z_2y_1 - z_1y_2 \\ z_1x_2 - z_2x_1 \\ x_1y_2 - y_1x_2 \end{pmatrix}$

which, restricted to  $\mathcal{C}$ , gives the commutative diagram :

$$\begin{array}{ccc}
 \mathcal{C} & \xrightarrow{(\Phi_{F,S})|_{\mathcal{C}}} & \Sigma_{\mathcal{S}}(\mathcal{C}) \\
 & \searrow R_{F,S} & \downarrow \delta_{\Sigma_{\mathcal{S}}(\mathcal{C})} \\
 & & \delta_{\Sigma_{\mathcal{S}}(\mathcal{C})}(\Sigma_{\mathcal{S}}(\mathcal{C})) \cong (\Sigma_{\mathcal{S}}(\mathcal{C}))^{\vee}
 \end{array} . \tag{8}$$

□

Let us notice that, according to the proof of Proposition 5, the rational map  $R_{F,S}$  has the same degree as the rational map  $\Phi_{F,S}$  (since  $\Sigma_{\mathcal{S}}(\mathcal{C})$  is irreducible and since the Gauss map  $(\delta_{\Sigma_{\mathcal{S}}(\mathcal{C})})|_{\Sigma_{\mathcal{S}}(\mathcal{C})}$  is birational [10]).

To compute the degree of  $\overline{R_{F,S}(\mathcal{C})}$ , we will use the fundamental lemma given in [14]. Let us first recall the definition of  $\varphi$ -polar introduced in [14] and extending the notion of polar.

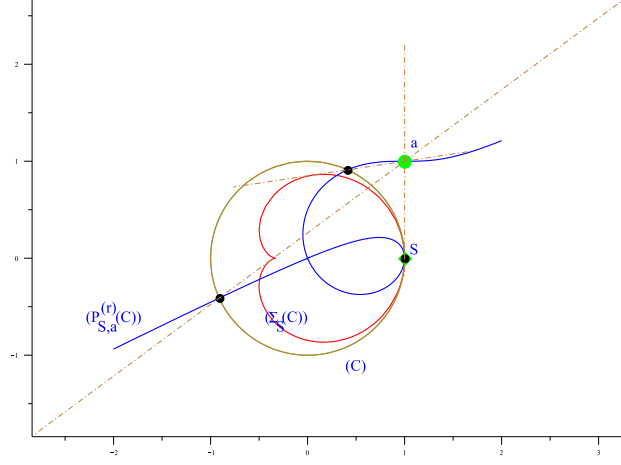
**Definition 6.** Let  $p \geq 1$ ,  $q \geq 1$ . Given  $\varphi : \mathbb{P}^p \rightarrow \mathbb{P}^q$  a rational map defined by  $\varphi = [\varphi_0 : \cdots : \varphi_q]$  (with  $\varphi_j \in \mathbb{C}[x_0, x_1, \dots, x_p]$  homogeneous of common degree  $d$ ) and  $a = [a_0 : \cdots : a_q] \in \mathbb{P}^q$ , we define the  $\varphi$ -polar at  $a$ , denoted by  $\mathcal{P}_{\varphi,a}$ , the hypersurface of degree  $d$  given by

$$\mathcal{P}_{\varphi,a} := V \left( \sum_{j=0}^q a_j \varphi_j \right) \subseteq \mathbb{P}^p.$$

With this definition, the “classical” polar of a curve  $\mathcal{C} = V(F)$  of  $\mathbb{P}^2$  (for some homogeneous polynomial  $F \in \mathbb{C}[x, y, z]$ ) at  $a$  is the  $\delta_{\mathcal{C}}$ -polar curve at  $a$ , where  $\delta_{\mathcal{C}} : [x : y : z] \mapsto [F_x : F_y : F_z]$ .

**Definition 7.** We call **reflected polar (or  $r$ -polar) of the plane curve  $\mathcal{C}$  with respect to  $\mathcal{S}$  at  $a$**  the  $R_{F,S}$ -polar at  $a$ , i.e. the curve  $\mathcal{P}_{\mathcal{S},a}^{(r)}(\mathcal{C}) := \mathcal{P}_{R_{F,S},a}$ .

From a geometric point of view,  $\mathcal{P}_{\mathcal{S},a}^{(r)}(\mathcal{C})$  is an algebraic curve such that, for every  $m \in \mathcal{C} \cap \mathcal{P}_{\mathcal{S},a}^{(r)}(\mathcal{C})$ ,  $R_m$  contains  $a$  (if  $R_m$  is well defined), this means that line  $(am)$  is tangent to  $\Sigma_{\mathcal{S}}(\mathcal{C})$  at the point  $m' = \Phi_{F,S}(m) \in \Sigma_{\mathcal{S}}(\mathcal{C})$  associated to  $m$  (see picture).



Let us now recall the statement of the fundamental lemma proved in [14].

**Lemma 2.1** (Fundamental lemma [14]). *Let  $\mathcal{C}$  be an irreducible algebraic curve of  $\mathbb{P}^p$  and  $\varphi : \mathbb{P}^p \rightarrow \mathbb{P}^q$  be a rational map given by  $\varphi = [\varphi_0 : \cdots : \varphi_q]$ , with  $\varphi_0, \dots, \varphi_q \in \mathbb{C}[x_0, \dots, x_p]$  some homogeneous polynomials of degree  $\delta$ . Assume that  $\mathcal{C} \not\subseteq \text{Base}(\varphi)$  and that  $\varphi|_{\mathcal{C}}$  has degree  $\delta_1 \in \mathbb{N} \cup \{\infty\}$ . Then, for generic  $a = [a_0 : \cdots : a_q] \in \mathbb{P}^q$ , the following formula holds true*

$$\delta_1 \cdot \deg(\overline{\varphi(\mathcal{C})}) = \delta \cdot \deg(\mathcal{C}) - \sum_{p \in \text{Base}(\varphi|_{\mathcal{C}})} i_p(\mathcal{C}, \mathcal{P}_{\varphi, a}),$$

with convention  $0 \cdot \infty = 0$  and  $\deg(\overline{\varphi(\mathcal{C})}) = 0$  if  $\#\overline{\varphi(\mathcal{C})} < \infty$ .

With the notations of this lemma, we define  $\text{mdeg}(\overline{\varphi(\mathcal{C})}) := \delta_1 \cdot \deg(\overline{\varphi(\mathcal{C})})$ , this quantity corresponds to the degree with multiplicity of  $\overline{\varphi(\mathcal{C})}$ . Thanks to this lemma, we get a first formula for the class (with multiplicity) of the caustic by reflection  $\Sigma_S(\mathcal{C})$ . Indeed we have

$$\text{mclass}(\Sigma_S(\mathcal{C})) = \text{mdeg}(\overline{R_{F,S}(\mathcal{C})}) = d(2d-1) - \sum_{m_1 \in \text{Base}((R_{F,S})|_{\mathcal{C}})} i_{m_1}(\mathcal{C}, \mathcal{P}_{S,a}^{(r)}(\mathcal{C})). \quad (9)$$

### 3. PROOF OF THEOREM 2

Formula (9) expresses the class of the caustic by reflection  $\Sigma_S(\mathcal{C})$  in terms of intersection numbers.

Now, we enter in the most technical stuff which is the computation of the intersection numbers  $i_{m_1}(\mathcal{C}, \mathcal{P}_{S,a}^{(r)}(\mathcal{C}))$  of  $\mathcal{C}$  with its reflected polar at the base points of  $R_{F,S}$ . To compute these



intersection numbers, it will be useful to observe the form of the image of  $R_{F,S}$  by linear changes of variable.

**Proposition 8.** *Let  $M \in GL(\mathbb{C}^3)$ . We have*

$$\tilde{R}_{F,S} \circ M = Com(M) \cdot \tilde{R}_{F \circ M, M^{-1}(S)}^{(M^{-1}(I), M^{-1}(J))},$$

with  $Com(M) := \det(M) \cdot {}^t M^{-1}$  and

$$\tilde{R}_{G,S'}^{(A,B)} := id \wedge [\Delta_A G \Delta_B G.S' - \Delta_{S'} G \Delta_A G.B - \Delta_{S'} G \Delta_B G.A].$$

*Proof.* This comes from the fact that  $M(U) \wedge M(V) = (Com(M))(U \wedge V)$  and  $\Delta_{M(U)}(F)(M(P)) = \Delta_U(F \circ M)(P)$ .  $\square$

Let  $m_1$  be a base point of  $\mathcal{C}$  and  $M \in GL(\mathbb{C}^3)$  be such that  $\Pi(M(0,0,1)) = m_1$  and such that the tangent cone of  $V(F \circ M)$  at  $[0:0:1]$  does not contain  $X = 0$ . Let  $\mu_{m_1}$  be the multiplicity of  $m_1$  in  $\mathcal{C}$ . We recall that  $m_1$  is a singular point of  $\mathcal{C}$  if and only if  $\mu_{m_1} > 1$ . Then, for every  $a \in \mathbb{P}^2$ , writing  $a' := \det(M)M^{-1}(a)$ , we have

$$\begin{aligned} i_{m_1}(\mathcal{C}, \mathcal{P}_{S,a}^{(r)}(\mathcal{C})) &= i_{m_1}(\mathcal{C}, V(\langle a, \tilde{R}_{F,S}(\cdot) \rangle)) \\ &= i_{[0:0:1]}(V(F \circ M), V(\langle a, \tilde{R}_{F,S} \circ M(\cdot) \rangle)) \\ &= i_{[0:0:1]}(V(F \circ M), V(\langle a', \tilde{R}_{F \circ M, M^{-1}(S)}^{(M^{-1}(I), M^{-1}(J))}(\cdot) \rangle)) \\ &= \sum_{\mathcal{B} \in Branch_{[0:0:1]}(V(F \circ M))} i_{[0:0:1]}(\mathcal{B}, V(\langle a', \tilde{R}_{F \circ M, M^{-1}(S)}^{(M^{-1}(I), M^{-1}(J))}(\cdot) \rangle)), \end{aligned}$$

where  $Branch_{[0:0:1]}(V(F \circ M))$  is the set of branches of  $V(F \circ M)$  at  $[0:0:1]$ . The last equality comes from Proposition 10 proved in appendix (see formula (13)). Let  $b$  be the number of such branches. Of course,  $b = 1$  for non-singular points. We write  $e_{\mathcal{B}}$  the multiplicity of the branch  $\mathcal{B}$ , we have  $\mu_{m_1} = \sum_{\mathcal{B} \in Branch_{[0:0:1]}(V(F \circ M))} e_{\mathcal{B}}$ . Now, as seen in [14], for every homogeneous polynomial  $G$ ,  $b$  and  $i_{[0:0:1]}(\mathcal{B}, V(G \circ M))$  do not depend on  $M$  chosen as above and we can adapt the choice of  $M$  to each of our branch.

Let us write  $\mathbb{C}\langle x^{\frac{1}{N}} \rangle$  and  $\mathbb{C}\langle x^{\frac{1}{N}}, y \rangle$  the rings of convergent power series of  $x^{\frac{1}{N}}, y$ . Let  $\mathbb{C}\langle x^* \rangle := \bigcup_{N \geq 1} \mathbb{C}\langle x^{\frac{1}{N}} \rangle$  and  $\mathbb{C}\langle x^*, y \rangle := \bigcup_{N \geq 1} \mathbb{C}\langle x^{\frac{1}{N}}, y \rangle$ . For every  $h = \sum_{q \in \mathbb{Q}} a_q x^q \in \mathbb{C}\langle x^* \rangle$ , we define the valuation of  $h$  as follows:

$$val(h) := val_x(h(x)) := \min\{q \in \mathbb{Q}, a_q \neq 0\}.$$

Let  $\mathcal{B}$  be a branch of  $V(F \circ M)$  at  $[0:0:1]$ . In the following,  $M$  is such that the tangent line to  $\mathcal{B}$  at  $[0:0:1]$  is  $Y = 0$ . Let  $A := M^{-1}(I)$ ,  $B := M^{-1}(J)$  and  $S' := M^{-1}(S)$ . Let  $\mathcal{T}_{\mathcal{B}}$  be the tangent line to  $\mathcal{B}$  at  $[0:0:1]$ . Branch  $\mathcal{B}$  can be splitted in  $e_{\mathcal{B}}$  pro-branches with equations  $y = g_{i,\mathcal{B}}(x)$  in the chart  $z = 1$  (for  $i \in \{1, \dots, e_{\mathcal{B}}\}$ ) with  $g_i \in \mathbb{C}\langle x^* \rangle$  having (rational) valuation larger than 1 (so  $g'_i(0) = 0$ ). For  $j \in \{1, \dots, e_{\mathcal{B}'}\}$ , consider also the equations  $y = g_{j,\mathcal{B}'}(x)$  (in the chart  $z = 1$ ) of the pro-branches  $\mathcal{V}_{j,\mathcal{B}'}$  for each branch  $\mathcal{B}' \in Branch_{[0:0:1]}(V(F \circ M))$ . This notion of pro-branches comes from the combination of the Weierstrass and the Puiseux theorems. It has been used namely by Halphen in [12] and by Wall in [19]. One can also see [14]. There exists a unit  $U$  of  $\mathbb{C}\langle x, y \rangle$  such that the following equality holds true in  $\mathbb{C}\langle x^*, y \rangle$

$$F(M(x, y, 1)) = U(x, y) \prod_{\mathcal{B}' \in Branch_{[0:0:1]}(V(F \circ M))} \prod_{j=1}^{e_{\mathcal{B}'}} (y - g_{j,\mathcal{B}'}(x)).$$

For a generic  $a$  (with  $a' := \det(M)M^{-1}(a)$ ), using the definition of intersection number of a branch with a curve (see [19] and (14)), we get

$$\begin{aligned} i_{[0:0:1]}(\mathcal{B}, V(\langle a', \tilde{R}_{F \circ M, S'}^{(A,B)}(\cdot) \rangle)) &= \sum_i \text{val}_x(\langle a', \tilde{R}_{F \circ M, S'}^{(A,B)}(x, g_{i,\mathcal{B}}(x), 1) \rangle) \\ &= \sum_i \min_{j=1,2,3} \text{val}_x \left( \left[ \tilde{R}_{F \circ M, M^{-1}(S)}^{(A,B)}(x, g_{i,\mathcal{B}}(x), 1) \right]_j \right). \end{aligned}$$

The quantity  $\text{val}_x \left( \left[ \tilde{R}_{F \circ M, M^{-1}(S)}^{(A,B)}(x, g_{i,\mathcal{B}}(x), 1) \right]_j \right)$  can be understood as the intersection number at  $[0 : 0 : 1]$  of pro-branch  $\mathcal{V}_{i,\mathcal{B}}$  with  $V \left( \left[ \tilde{R}_{F \circ M, M^{-1}(S)}^{(A,B)} \right]_j \right)$ . Observe that, for every  $m_1 \in \mathcal{C}$ , we have

$$\begin{aligned} m_1 \in \text{Base}(R_{F,S}) &\Leftrightarrow \tilde{R}_{F \circ M, M^{-1}(S)}^{(A,B)}(0, 0, 1) = (0, 0, 0) \\ &\Leftrightarrow \sum_{i=1}^{e_{\mathcal{B}}} \min_{j=1,2,3} \text{val}_x \left( \left[ \tilde{R}_{F \circ M, M^{-1}(S)}^{(A,B)}(x, g_{i,\mathcal{B}}(x), 1) \right]_j \right) > 0 \\ &\Leftrightarrow \sum_{\mathcal{B}'} \sum_{i=1}^{e_{\mathcal{B}'}} \min_{j=1,2,3} \text{val}_x \left( \left[ \tilde{R}_{F \circ M, M^{-1}(S)}^{(A,B)}(x, g_{i,\mathcal{B}}(x), 1) \right]_j \right) > 0. \end{aligned}$$

Therefore, Formula (9) becomes

$$\text{mclass}(\Sigma_{\mathcal{S}}(\mathcal{C})) := d(2d-1) - \sum_{m_1 \in \mathcal{C}} \sum_{\mathcal{B}} \sum_{i=1}^{e_{\mathcal{B}}} \min_{j=1,2,3} \text{val}_x \left( \left[ \tilde{R}_{F \circ M, M^{-1}(S)}^{(A,B)}(x, g_{i,\mathcal{B}}(x), 1) \right]_j \right), \quad (10)$$

where, for every  $m_1 \in \mathcal{C}$ ,  $M$  depends on  $m_1$  and is as above, where the sum is over  $\mathcal{B} \in \text{Branch}_{[0:0:1]}(V(F \circ M))$ .

It will be useful to notice that, for every  $P = (x_P, y_P, z_P) \in \mathbb{C}^3 \setminus \{0\}$ , we have

$$(\Delta_{M(P)}F) \circ M(x, g_{i,\mathcal{B}}(x), 1) = \Delta_P(F \circ M)(x, g_{i,\mathcal{B}}(x), 1) = D_{i,\mathcal{B}}(x)W_{P,i,\mathcal{B}}(x),$$

with

$$W_{P,i,\mathcal{B}}(x) := y_P - g'_{i,\mathcal{B}}(x)x_P + z_P(xg'_{i,\mathcal{B}}(x) - g_{i,\mathcal{B}}(x))$$

and with  $D_{i,\mathcal{B}}(x) := U(x, g_{i,\mathcal{B}}(x)) \prod_{\mathcal{B}' \in \text{Branch}_{[0:0:1]}(V(F \circ M))} \prod_{j=1, \dots, e_{\mathcal{B}'}: (\mathcal{B}', j) \neq (\mathcal{B}, i)} (g_{i,\mathcal{B}}(x) - g_{j,\mathcal{B}'}(x))$ . Hence we have

$$\tilde{R}_{F \circ M, S'}^{(A,B)}(x, g_{i,\mathcal{B}}(x), 1) := (D_{i,\mathcal{B}})^2 \cdot \hat{R}_{i,\mathcal{B}}(x), \quad (11)$$

with

$$\hat{R}_{i,\mathcal{B}}(x) := \begin{pmatrix} x \\ g_{i,\mathcal{B}}(x) \\ 1 \end{pmatrix} \wedge [W_{A,i,\mathcal{B}}(x)W_{B,i,\mathcal{B}}(x).S' - W_{S',i,\mathcal{B}}(x)W_{A,i,\mathcal{B}}(x).B - W_{S',i,\mathcal{B}}(x)W_{B,i,\mathcal{B}}(x).A].$$

Let us write

$$h_{m_1,i,\mathcal{B}} := \min(\text{val}([\hat{R}_{i,\mathcal{B}}]_j), j = 1, 2, 3) \quad \text{and} \quad h_{m_1} = \sum_{\mathcal{B} \in \text{Branch}_{[0:0:1]}(V(F \circ M))} \sum_{i=1}^{e_{\mathcal{B}}} h_{m_1,i,\mathcal{B}}.$$

With the notations of [14] (since  $U(0,0) \neq 0$ ), we have

$$\sum_{\mathcal{B} \in \text{Branch}_{[0:0:1]}(V(F \circ M))} \sum_{i=1}^{e_{\mathcal{B}}} \text{val}(D_{i,\mathcal{B}}) = V_{m_1},$$

which is null if  $m_1$  is a non-singular point of  $\mathcal{C}$ . With these notations, thanks to (11), formula (10) becomes

$$mclass(\Sigma_{\mathcal{S}}(\mathcal{C})) = 2d(d-1) + d - 2 \sum_{m_1 \in Sing(\mathcal{C})} V_{m_1} - \sum_{m_1 \in \mathcal{C}} h_{m_1}.$$

Moreover, as noticed in [14], we have

$$d(d-1) - \sum_{m_1 \in Sing(\mathcal{C})} V_{m_1} = d^{\vee},$$

where  $d^{\vee}$  is the class of  $\mathcal{C}$ . Therefore, we get

$$mclass(\Sigma_{\mathcal{S}}(\mathcal{C})) = 2d^{\vee} + d - \sum_{m_1 \in \mathcal{C}} h_{m_1}.$$

Theorem 2 comes now directly from the computation of  $h_{m_1, i, \mathcal{B}}$  given in following result.

**Lemma 9.** *Let  $m_1 \in \mathcal{C}$ . We have*

- $\sum_{i=1}^{e_{\mathcal{B}}} h_{m_1, i, \mathcal{B}} = 0$  if  $\mathcal{I}, \mathcal{J}, \mathcal{S} \notin \mathcal{T}_{\mathcal{B}}$ .
- $\sum_{i=1}^{e_{\mathcal{B}}} h_{m_1, i, \mathcal{B}} = 0$  if  $\#(\mathcal{T}_{\mathcal{B}} \cap \{\mathcal{I}, \mathcal{J}, \mathcal{S}\}) = 1$  and  $m_1 \notin \{\mathcal{I}, \mathcal{J}, \mathcal{S}\}$ .
- $\sum_{i=1}^{e_{\mathcal{B}}} h_{m_1, i, \mathcal{B}} = e_{\mathcal{B}}$  if  $\#(\mathcal{T}_{\mathcal{B}} \cap \{\mathcal{I}, \mathcal{J}, \mathcal{S}\}) = 1$  and  $m_1 \in \{\mathcal{I}, \mathcal{J}, \mathcal{S}\}$ .
- $\sum_{i=1}^{e_{\mathcal{B}}} h_{m_1, i, \mathcal{B}} = i_{m_1}(\mathcal{B}, \mathcal{T}_{\mathcal{B}}) + \min(i_{m_1}(\mathcal{B}, \mathcal{T}_{\mathcal{B}}) - 2e_{\mathcal{B}}, 0)$  if  $\mathcal{T}_{\mathcal{B}} = (\mathcal{I}\mathcal{S})$ ,  $\mathcal{J} \notin \mathcal{T}_{\mathcal{B}}$  and  $m_1 \notin \{\mathcal{I}, \mathcal{S}\}$ .  
 $\sum_{i=1}^{e_{\mathcal{B}}} h_{m_1, i, \mathcal{B}} = i_{m_1}(\mathcal{B}, \mathcal{T}_{\mathcal{B}}) + \min(i_{m_1}(\mathcal{B}, \mathcal{T}_{\mathcal{B}}) - 2e_{\mathcal{B}}, 0)$  if  $\mathcal{T}_{\mathcal{B}} = (\mathcal{J}\mathcal{S})$ ,  $\mathcal{I} \notin \mathcal{T}_{\mathcal{B}}$  and  $m_1 \notin \{\mathcal{J}, \mathcal{S}\}$ .
- $\sum_{i=1}^{e_{\mathcal{B}}} h_{m_1, i, \mathcal{B}} = i_{m_1}(\mathcal{B}, \mathcal{T}_{\mathcal{B}})$  if  $\mathcal{T}_{\mathcal{B}} = (\mathcal{I}\mathcal{S})$ ,  $\mathcal{J} \notin \mathcal{T}_{\mathcal{B}}$  and  $m_1 \in \{\mathcal{I}, \mathcal{S}\}$ .  
 $\sum_{i=1}^{e_{\mathcal{B}}} h_{m_1, i, \mathcal{B}} = i_{m_1}(\mathcal{B}, \mathcal{T}_{\mathcal{B}})$  if  $\mathcal{T}_{\mathcal{B}} = (\mathcal{J}\mathcal{S})$ ,  $\mathcal{I} \notin \mathcal{T}_{\mathcal{B}}$  and  $m_1 \in \{\mathcal{J}, \mathcal{S}\}$ .
- $\sum_{i=1}^{e_{\mathcal{B}}} h_{m_1, i, \mathcal{B}} = i_{m_1}(\mathcal{B}, \mathcal{T}_{\mathcal{B}}) - e_{\mathcal{B}}$  if  $\mathcal{T}_{\mathcal{B}} = (\mathcal{I}\mathcal{J})$ , that  $\mathcal{S} \notin \mathcal{T}_{\mathcal{B}}$  and  $m_1 \notin \{\mathcal{I}, \mathcal{J}\}$ .
- $\sum_{i=1}^{e_{\mathcal{B}}} h_{m_1, i, \mathcal{B}} = i_{m_1}(\mathcal{B}, \mathcal{T}_{\mathcal{B}})$  if  $\mathcal{T}_{\mathcal{B}} = (\mathcal{I}\mathcal{J})$ , that  $\mathcal{S} \notin \mathcal{T}_{\mathcal{B}}$  and  $m_1 \in \{\mathcal{I}, \mathcal{J}\}$ .
- $\sum_{i=1}^{e_{\mathcal{B}}} h_{m_1, i, \mathcal{B}} = 2i_{m_1}(\mathcal{B}, \mathcal{T}_{\mathcal{B}}) - 2e_{\mathcal{B}}$  if  $\mathcal{I}, \mathcal{J}, \mathcal{S} \in \mathcal{T}_{\mathcal{B}}$  and  $m_1 \notin \{\mathcal{I}, \mathcal{J}, \mathcal{S}\}$ .
- $\sum_{i=1}^{e_{\mathcal{B}}} h_{m_1, i, \mathcal{B}} = 2i_{m_1}(\mathcal{B}, \mathcal{T}_{\mathcal{B}}) - e_{\mathcal{B}}$  if  $\mathcal{I}, \mathcal{J}, \mathcal{S} \in \mathcal{T}_{\mathcal{B}}$  and  $m_1 \in \{\mathcal{I}, \mathcal{J}\}$ .
- $\sum_{i=1}^{e_{\mathcal{B}}} h_{m_1, i, \mathcal{B}} = 2i_{m_1}(\mathcal{B}, \mathcal{T}_{\mathcal{B}}) - e_{\mathcal{B}}$  if  $\mathcal{I}, \mathcal{J}, \mathcal{S} \in \mathcal{T}_{\mathcal{B}}$ ,  $m_1 = \mathcal{S}$  and  $i_{m_1}(\mathcal{B}, \mathcal{T}_{\mathcal{B}}) \neq 2e_{\mathcal{B}}$ .
- $\sum_{i=1}^{e_{\mathcal{B}}} h_{m_1, i, \mathcal{B}} = e_{\mathcal{B}}(1 + \min(\beta_1, 3))$  if  $\mathcal{I}, \mathcal{J}, \mathcal{S} \in \mathcal{T}_{\mathcal{B}}$ ,  $m_1 = \mathcal{S}$  and  $i_{m_1}(\mathcal{B}, \mathcal{T}_{\mathcal{B}}) = 2e_{\mathcal{B}}$ , where  $\beta_1$  is the degree of the lowest degree term of  $g_{1, \mathcal{B}}$  of degree larger than 2, i.e.  $e_{\mathcal{B}}\beta_1 = i_{m_1}(\mathcal{B}, Osc_{m_1}(\mathcal{B}))$ , where  $Osc_{m_1}(\mathcal{B})$  is any osculating smooth algebraic curve to  $\mathcal{B}$  at  $m_1$  (which is well-defined since  $i_{m_1}(\mathcal{B}, \mathcal{T}_{\mathcal{B}}) \geq 2e_{\mathcal{B}}$ ).

Observe that the points  $m_1 \in \mathcal{C}$  that are not base points of  $R_{F, \mathcal{S}}$  are the non-singular points of  $\mathcal{C}$  such that  $h_{m_1} = 0$ .

*Proof.* To simplify notations, we omit indices  $\mathcal{B}$  and consider  $i \in \{1, \dots, e_{\mathcal{B}}\}$ .

- (1) Suppose that  $\mathcal{I}, \mathcal{J}, \mathcal{S} \notin \mathcal{T}_{\mathcal{B}}$ .

Then  $W_{B, i}(0) = y_B \neq 0$ ,  $W_{A, i}(0) = y_A \neq 0$  and  $W_{S', i}(0) = y_{S'} \neq 0$  so

$$\begin{aligned} \hat{R}_i(0) &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \wedge [y_A y_B \cdot S' - y_A y_{S'} \cdot B - y_B y_{S'} \cdot A] \\ &= \begin{pmatrix} y_A y_B y_{S'} \\ y_A y_B x_{S'} - y_A y_{S'} x_B - y_B y_{S'} x_A \\ 0 \end{pmatrix}. \end{aligned}$$

Hence  $h_{m_1, i, \mathcal{B}} = 0$  and the sum over  $i = 1, \dots, e_{\mathcal{B}}$  of these quantities is equal to 0.

- (2) Suppose that  $\mathcal{I} \in \mathcal{T}_B$ ,  $\mathcal{J}, \mathcal{S} \notin \mathcal{T}_B$  and  $m_1 \neq \mathcal{I}$ .

Take  $M$  such that  $S' = (0, 1, 0)$ ,  $A = (1, 0, 0)$ ,  $y_B \neq 0$ . We have  $W_{B,i}(0) = y_B$ ,  $W_{A,i}(0) = 0$  and  $W_{S',i}(0) = 1$  and so

$$\hat{R}_i(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \wedge \begin{pmatrix} -y_B \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -y_B \\ 0 \end{pmatrix}.$$

Hence  $h_{m_1,i,B} = 0$  and the sum over  $i = 1, \dots, e_B$  of these quantities is equal to 0.

- (3) Suppose that  $\mathcal{I} \in \mathcal{T}_B$ ,  $\mathcal{J}, \mathcal{S} \notin \mathcal{T}_B$  and  $m_1 = \mathcal{I}$ .

Take  $M$  such that  $S' = (0, 1, 0)$ ,  $A = (0, 0, 1)$ ,  $y_B \neq 0$ . We have  $W_{B,i}(x) = y_B - g'_i(x)x_B + z_B(xg'_i(x) - g_i(x))$ ,  $W_{A,i}(x) = xg'_i(x) - g_i(x)$  and  $W_{S',i}(x) = 1$  and so

$$\begin{aligned} \hat{R}_i(x) &= \begin{pmatrix} x \\ g_i(x) \\ 1 \end{pmatrix} \wedge \begin{pmatrix} -(xg'_i(x) - g_i)x_B \\ (xg'_i(x) - g_i)(-g'_i(x)x_B + z_B(xg'_i(x) - g_i(x))) \\ -y_B + g'_i(x)x_B - 2z_B(xg'_i(x) - g_i(x)) \end{pmatrix} \\ &= \begin{pmatrix} -y_B g_i(x) + x(g'_i(x))^2 x_B - z_B((xg'_i(x))^2 - (g_i(x))^2) \\ -x_B(2xg'_i(x) - g_i(x)) + xy_B + 2xz_B(xg'_i(x) - g_i(x)) \\ -x_B(xg'_i(x) - g_i(x))^2 + z_B(xg'_i(x) - g_i(x)) \end{pmatrix}, \end{aligned}$$

the valuation of the coordinates of which are larger than or equal to 1 and the valuation of the second coordinate is 1.

Hence  $h_{m_1,i,B} = 1$  and the sum over  $i = 1, \dots, e_B$  of these quantities is equal to  $e_B$ .

- (4) Suppose that  $\mathcal{S} \in \mathcal{T}_B$ ,  $\mathcal{I}, \mathcal{J} \notin \mathcal{T}_B$  and  $m_1 \neq \mathcal{S}$ .

Take  $M$  such that  $A = (0, 1, 0)$ ,  $S' = (1, 0, 0)$ ,  $y_B \neq 0$ . We have  $W_{B,i}(0) = y_B \neq 0$ ,  $W_{S',i}(0) = 0$  and  $W_{A,i}(0) = 1$  and so

$$\hat{R}_i(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \wedge \begin{pmatrix} y_B \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ y_B \\ 0 \end{pmatrix}.$$

Hence  $h_{m_1,i,B} = 0$  and the sum over  $i = 1, \dots, e_B$  of these quantities is equal to 0.

- (5) Suppose that  $m_1 = \mathcal{S}$  and  $\mathcal{I}, \mathcal{J} \notin \mathcal{T}_B$ .

Take  $M$  such that  $S' = (0, 0, 1)$ ,  $A = (0, 1, 0)$ ,  $y_B \neq 0$ . We have  $W_{B,i}(x) = y_B - g'_i(x)x_B + z_B(xg'_i(x) - g_i(x))$ ,  $W_{S',i}(x) = xg'_i(x) - g_i(x)$  and  $W_{A,i}(x) = 1$  and so

$$\begin{aligned} \hat{R}_i(x) &= \begin{pmatrix} x \\ g_i(x) \\ 1 \end{pmatrix} \wedge \begin{pmatrix} -(xg'_i(x) - g_i)x_B \\ -(xg'_i(x) - g_i)(2y_B - g'_i(x)x_B + z_B(xg'_i(x) - g_i(x))) \\ y_B - g'_i(x)x_B \end{pmatrix} \\ &= \begin{pmatrix} g_i(x)(y_B - g'_i(x)x_B) + (xg'_i(x) - g_i(x))(2y_B - g'_i(x)x_B + z_B(xg'_i(x) - g_i(x))) \\ -(xg'_i(x) - g_i(x))x_B - x(y_B - g'_i(x)x_B) \\ -(xg'_i(x) - g_i(x))(2xy_B - g'_i(x)x_B + z_Bx(xg'_i(x) - g_i(x))) + g_i(x)(xg'_i(x) - g_i(x))x_B \end{pmatrix}, \end{aligned}$$

the valuation of the coordinates of which are larger than or equal to 1 and the valuation of the second coordinate is 1.

Hence  $h_{m_1,i,B} = 1$  and the sum over  $i = 1, \dots, e_B$  of these quantities is equal to  $e_B$ .

- (6) Suppose that  $\mathcal{T}_B = (\mathcal{IS})$ , that  $\mathcal{J} \notin \mathcal{T}_B$  and  $m_1 \notin \{\mathcal{I}, \mathcal{S}\}$ .

Take  $M$  such that  $S' = (1, 0, 0)$ ,  $B = (0, 1, 0)$ ,  $y_A = 0$ ,  $x_A \neq 0$ ,  $z_A \neq 0$ . We have  $W_{S',i}(x) = -g'_i(x)$ ,  $W_{A,i}(x) = -g'_i(x)x_A + z_A(xg'_i(x) - g_i(x))$  and  $W_{B,i}(x) = 1$  and so

$$\begin{aligned}\hat{R}_i(x) &= \begin{pmatrix} x \\ g_i(x) \\ 1 \end{pmatrix} \wedge \begin{pmatrix} z_A(xg'_i(x) - g_i(x)) \\ -(g'_i(x))^2x_A + g'_i(x)(xg'_i(x) - g_i(x))z_A \\ g'_i(x)z_A \end{pmatrix} \\ &= \begin{pmatrix} g_i(x)g'_i(x)z_A + (g'_i(x))^2x_A - g'_i(x)(xg'_i(x) - g_i(x))z_A \\ -g_i(x)z_A \\ -x(g'_i(x))^2x_A + (xg'_i(x) - g_i(x))^2z_A \end{pmatrix},\end{aligned}$$

the valuation of the coordinates of which are respectively  $2\text{val}(g_i) - 2$ ,  $\text{val}(g_i)$  and  $2\text{val}(g_i) - 1$ .

Hence  $h_{m_1,i,B} = \text{val}(g_i) + \min(\text{val}(g_i) - 2, 0)$  and the sum over  $i = 1, \dots, e_B$  of these quantities is equal to  $i_{m_1}(\mathcal{B}, \mathcal{T}_B) + \min(i_{m_1}(\mathcal{B}, \mathcal{T}_B) - 2e_B, 0)$ .

- (7) Suppose that  $\mathcal{T}_B = (\mathcal{I}\mathcal{S})$ , that  $\mathcal{J} \notin \mathcal{T}_B$  and  $m_1 = \mathcal{I}$ .

Take  $M$  such that  $S' = (1, 0, 0)$ ,  $B = (0, 1, 0)$ ,  $A = (0, 0, 1)$ . We have  $W_{S',i}(x) = -g'_i(x)$ ,  $W_{A,i}(x) = xg'_i(x) - g_i(x)$  and  $W_{B,i}(x) = 1$  and so

$$\begin{aligned}\hat{R}_i(x) &= \begin{pmatrix} x \\ g_i(x) \\ 1 \end{pmatrix} \wedge \begin{pmatrix} xg'_i(x) - g_i(x) \\ g'_i(x)(xg'_i(x) - g_i(x)) \\ g'_i(x) \end{pmatrix} \\ &= \begin{pmatrix} g'_i(x)(2g_i(x) - xg'_i(x)) \\ -g_i(x) \\ (xg'_i(x) - g_i(x))^2 \end{pmatrix},\end{aligned}$$

the valuation of the coordinates of which are larger than or equal to  $\text{val}(g_i)$ , the second coordinate has valuation  $\text{val}(g_i)$ .

Hence  $h_{m_1,i,B} = \text{val}(g_i)$  and the sum over  $i = 1, \dots, e_B$  of these quantities is equal to  $i_{m_1}(\mathcal{B}, \mathcal{T}_B)$ .

- (8) Suppose that  $\mathcal{T}_B = (\mathcal{I}\mathcal{S})$ , that  $\mathcal{J} \notin \mathcal{T}_B$  and  $m_1 = \mathcal{S}$ .

Take  $M$  such that  $A = (1, 0, 0)$ ,  $B = (0, 1, 0)$ ,  $S' = (0, 0, 1)$ . We have  $W_{S',i}(x) = xg'_i(x) - g_i(x)$ ,  $W_{A,i}(x) = -g'_i(x)$  and  $W_{B,i}(x) = 1$  and so

$$\begin{aligned}\hat{R}_i(x) &= \begin{pmatrix} x \\ g_i(x) \\ 1 \end{pmatrix} \wedge \begin{pmatrix} -(xg'_i(x) - g_i(x)) \\ g'_i(x)g'_i(x) - g_i(x) \\ -g'_i(x) \end{pmatrix} \\ &= \begin{pmatrix} -x(g'_i(x))^2 \\ g_i(x) \\ (xg'_i(x))^2 - (g_i(x))^2 \end{pmatrix},\end{aligned}$$

the valuation of the coordinates of which being larger than or equal to  $\text{val}(g_i)$  and the valuation of the second coordinate is equal to  $\text{val}(g_i)$ .

Hence  $h_{m_1,i,B} = \text{val}(g_i)$  and the sum over  $i = 1, \dots, e_B$  of these quantities is equal to  $i_{m_1}(\mathcal{B}, \mathcal{T}_B)$ .

- (9) Suppose that  $\mathcal{T}_B = (\mathcal{I}\mathcal{J})$ , that  $\mathcal{S} \notin \mathcal{T}_B$  and  $m_1 \notin \{\mathcal{I}, \mathcal{J}\}$ .

Take  $M$  such that  $S' = (0, 1, 0)$ ,  $B = (1, 0, 0)$ ,  $y_A = 0$ ,  $x_A \neq 0$ ,  $z_A \neq 0$ . We have  $W_{B,i}(x) = -g'_i(x)$ ,  $W_{A,i}(x) = -g'_i(x)x_A + z_A(xg'_i(x) - g_i(x))$  and  $W_{S',i}(x) = 1$  and so

$$\begin{aligned}\hat{R}_i(x) &= \begin{pmatrix} x \\ g_i(x) \\ 1 \end{pmatrix} \wedge \begin{pmatrix} 2g'_i(x)x_A - z_A(xg'_i(x) - g_i(x)) \\ (g'_i(x))^2x_A - g'_i(x)(xg'_i(x) - g_i(x))z_A \\ g'_i(x)z_A \end{pmatrix} \\ &= \begin{pmatrix} (g'_i(x))^2(xz_A - x_A) \\ 2g'_i(x)x_A - z_A(2xg'_i(x) - g_i(x)) \\ -z_A(xg'_i(x) - g_i(x))^2 + x_Ag'_i(x)(xg'_i(x) - 2g_i(x)) \end{pmatrix},\end{aligned}$$

the valuation of the coordinates of which are respectively  $2\text{val}(g_i) - 2$ ,  $\text{val}(g_i) - 1$  and larger than  $\text{val}(g_i)$ . Hence  $h_{m_1,i,B} = \text{val}(g_i) - 1$  and the sum over  $i = 1, \dots, e_B$  of these quantities is equal to  $i_{m_1}(\mathcal{B}, \mathcal{T}_B) - e_B$ .

- (10) Suppose that  $\mathcal{T}_B = (\mathcal{I}\mathcal{J})$ , that  $\mathcal{S} \notin \mathcal{T}_B$  and  $m_1 = \mathcal{I}$ .

Take  $M$  such that  $S' = (0, 1, 0)$ ,  $B = (1, 0, 0)$ ,  $A = (0, 0, 1)$ . We have  $W_{B,i}(x) = -g'_i(x)$ ,  $W_{A,i}(x) = xg'_i(x) - g_i(x)$  and  $W_{S',i}(x) = 1$  and so

$$\begin{aligned}\hat{R}_i(x) &= \begin{pmatrix} x \\ g_i(x) \\ 1 \end{pmatrix} \wedge \begin{pmatrix} -(xg'_i(x) - g_i(x)) \\ -g'_i(x)(xg'_i(x) - g_i(x)) \\ g'_i(x) \end{pmatrix} \\ &= \begin{pmatrix} x(g'_i(x))^2 \\ -(2xg'_i(x) - g_i(x)) \\ -(xg'_i(x) - g_i(x))^2 \end{pmatrix},\end{aligned}$$

the valuation of the coordinates of which being larger than or equal to  $\text{val}(g_i)$  and the valuation of the second coordinate is equal to  $\text{val}(g_i)$ .

Hence  $h_{m_1,i,B} = \text{val}(g_i)$  and the sum over  $i = 1, \dots, e_B$  of these quantities is equal to  $i_{m_1}(\mathcal{B}, \mathcal{T}_B)$ .

- (11) Suppose that  $\mathcal{I}, \mathcal{J}, \mathcal{S} \in \mathcal{T}_B$  and  $m_1 \notin \{\mathcal{I}, \mathcal{J}, \mathcal{S}\}$ .

Take  $M$  such that  $S' = (1, 0, 0)$ ,  $y_A = y_B = 0$ ,  $x_A \neq 0$ ,  $z_A \neq 0$ ,  $x_B \neq 0$ ,  $z_B \neq 0$ ,  $x_A z_B \neq x_B z_A$ .

We have  $W_{S',i}(x) = -g'_i(x)$ ,  $W_{A,i}(x) = -g'_i(x)x_A + z_A(xg'_i(x) - g_i(x))$  and  $W_{B,i}(x) = -g'_i(x)x_B + z_B(xg'_i(x) - g_i(x))$  and so

$$\begin{aligned}\hat{R}_i(x) &= \begin{pmatrix} x \\ g_i(x) \\ 1 \end{pmatrix} \wedge \begin{pmatrix} -x_A(g'_i(x))^2x_B + z_A z_B(xg'_i(x) - g_i(x))^2 \\ 0 \\ -(g'_i(x))^2(xz_B + x_B z_A) + 2z_A z_B g'_i(x)(xg'_i(x) - g_i(x)) \end{pmatrix} \\ &= \begin{pmatrix} -g_i(x)(g'_i(x))^2(xz_B + x_B z_A) + 2z_A z_B g_i(x)g'_i(x)(xg'_i(x) - g_i(x)) \\ -x_A(g'_i(x))^2x_B + z_A z_B(xg'_i(x) - g_i(x))^2 - x[\dots] \\ x_A g_i(x)(g'_i(x))^2x_B - z_A z_B g_i(x)(xg'_i(x) - g_i(x))^2 \end{pmatrix},\end{aligned}$$

the valuation of the coordinates of which are larger than or equal to  $2\text{val}(g_i) - 2$ , the valuation of the second coordinate is  $2\text{val}(g_i) - 2$ . Hence  $h_{m_1,i,B} = 2\text{val}(g_i) - 2$  and the sum over  $i = 1, \dots, e_B$  of these quantities is equal to  $2i_{m_1}(\mathcal{B}, \mathcal{T}_B) - 2e_B$ .

- (12) Suppose that  $\mathcal{I}, \mathcal{J}, \mathcal{S} \in \mathcal{T}_B$  and  $m_1 = \mathcal{J}$ .

Take  $M$  such that  $B = (0, 0, 1)$ ,  $S' = (1, 0, 0)$ ,  $y_A = 0$ ,  $x_A \neq 0$  and  $z_A \neq 0$ . We have  $W_{S',i}(x) = -g'_i(x)$ ,  $W_{A,i}(x) = -g'_i(x)x_A + z_A(xg'_i(x) - g_i(x))$  and  $W_{B,i}(x) = xg'_i(x) - g_i(x)$

and so

$$\begin{aligned}\hat{R}_i(x) &= \begin{pmatrix} x \\ g_i(x) \\ 1 \end{pmatrix} \wedge \begin{pmatrix} z_A(xg'_i(x) - g_i(x))^2 \\ 0 \\ -x_A(g'_i(x))^2 + 2z_A(xg'_i(x) - g_i(x))g'_i(x) \end{pmatrix} \\ &= \begin{pmatrix} g_i(x)g'_i(x)(-g'_i(x)x_A + 2z_A(xg'_i(x) - g_i(x))) \\ z_A(xg'_i(x) - g_i(x))^2 - xg'_i(x)(-g'_i(x)x_A + 2z_A(xg'_i(x) - g_i(x))) \\ -g_i(x)z_A(xg'_i(x) - g_i(x))^2 \end{pmatrix},\end{aligned}$$

the valuation of the coordinates of which are larger than or equal to  $2\text{val}(g_i) - 1$  and the valuation of the second coordinate is  $2\text{val}(g_i) - 1$ .

Hence  $h_{m_1,i,\mathcal{B}} = 2\text{val}(g_i) - 1$  and the sum over  $i = 1, \dots, e_{\mathcal{B}}$  of these quantities is equal to  $2i_{m_1}(\mathcal{B}, \mathcal{T}_{\mathcal{B}}) - e_{\mathcal{B}}$ .

(13) Suppose that  $\mathcal{I}, \mathcal{J}, \mathcal{S} \in \mathcal{T}_{\mathcal{B}}$  and  $m_1 = \mathcal{S}$ .

Take  $M$  such that  $S' = (0, 0, 1)$ ,  $B = (1, 0, 0)$ ,  $y_A = 0$ ,  $x_A \neq 0$  and  $z_A \neq 0$ . We have  $W_{B,i}(x) = -g'_i(x)$ ,  $W_{A,i}(x) = -g'_i(x)x_A + z_A(xg'_i(x) - g_i(x))$  and  $W_{S',i}(x) = xg'_i(x) - g_i(x)$  and so

$$\begin{aligned}\hat{R}_i(x) &= \begin{pmatrix} x \\ g_i(x) \\ 1 \end{pmatrix} \wedge \begin{pmatrix} 2x_Ag'_i(x)(xg'_i(x) - g_i(x)) - z_A(xg'_i(x) - g_i(x))^2 \\ 0 \\ (g'_i(x))^2x_A \end{pmatrix} \\ &= \begin{pmatrix} g_i(x)(g'_i(x))^2x_A \\ x_Ag'_i(x)(xg'_i(x) - 2g_i(x)) - z_A(xg'_i(x) - g_i(x))^2 \\ -2x_Ag_i(x)g'_i(x)(xg'_i(x) - g_i(x)) + z_Ag_i(x)(xg'_i(x) - g_i(x))^2 \end{pmatrix}.\end{aligned}$$

The valuation of the first coordinate is  $3\text{val}(g_i) - 2$  is smaller than or equal to the valuation of the third coordinate.

Now, if  $\text{val}(g_i) \neq 2$ , the valuation of the second coordinate is  $2\text{val}(g_i) - 1$ ; hence  $h_{m_1,i,\mathcal{B}} = 2\text{val}(g_i) - 1$  and the sum over  $i = 1, \dots, e_{\mathcal{B}}$  of these quantities is equal to  $2i_{m_1}(\mathcal{B}, \mathcal{T}_{\mathcal{B}}) - e_{\mathcal{B}}$ .

Suppose now that  $\text{val}(g_i) = 2$ , then  $3\text{val}(g_i) - 2 = 4$  and there exist  $\alpha, \alpha_1 \in \mathbb{C}$  and  $\beta_1 > 2$  such that  $g_i(x) = \alpha x^2 + \alpha_1 x^{\beta_1} + \dots$ . Then, the second coordinate has the following form

$$(x_A 2\alpha(\beta_1 - 2)x^{\beta_1+1} + \dots) + x^4(\dots).$$

Therefore  $h_{m_1,i,\mathcal{B}} = \min(\beta_1 + 1, 4)$  and the sum over  $i = 1, \dots, e_{\mathcal{B}}$  of these quantities is equal to  $e_{\mathcal{B}}(1 + \min(\beta_1, 3))$ .

□

#### 4. PROOF OF PROPOSITION 1

Observe that  $\Sigma_{\mathcal{I}}(\mathcal{C}) = \{\mathcal{J}\}$  and  $\Sigma_{\mathcal{J}}(\mathcal{C}) = \{\mathcal{I}\}$ .

Assume (i), (ii) and (iii) and that  $\Sigma_{\mathcal{S}}(\mathcal{C}) = \{\mathcal{S}'\}$  with  $\mathcal{S}' = [x_1 : y_1 : z_1]$ .

When  $\mathcal{S} \notin \ell_{\infty}$ , we will use the fact that  $\Sigma_{\mathcal{S}}(\mathcal{C})$  is the evolute of the orthotomic of  $\mathcal{C}$  with respect to  $\mathcal{S}$ . Since  $\mathcal{C}$  is not a line, the orthotomic of  $\mathcal{C}$  with respect to  $\mathcal{S}$  is not reduced to a point but its evolute is a point. This implies that the orthotomic of  $\mathcal{C}$  with respect to  $\mathcal{S}$  is either a line (not equal to  $\ell_{\infty}$ ) or a circle. But  $\mathcal{C}$  is the contrapedal (or orthocaustic) curve (from  $\mathcal{S}$ ) of the image by the  $\mathcal{S}$ -centered homothety (with ratio  $1/2$ ) of the orthotomic of  $\mathcal{C}$ . Therefore  $d = 2$  and  $\mathcal{S}$  is a focal point of  $\mathcal{C}$ , which contradicts (iii).

When  $\mathcal{S} \in \ell_{\infty}$  but  $\mathcal{S}' \notin \ell_{\infty}$ , then, for symmetry reasons, we also have  $\Sigma_{\mathcal{S}'}(\mathcal{C}) = \{\mathcal{S}\}$  and we conclude analogously.

Suppose now that  $\mathcal{S}, \mathcal{S}' \in \ell_\infty$ . We have  $z_0 = z_1 = 0$ . For every  $m = [x : y : 1] \in \mathcal{C} \setminus \ell_\infty$ , we have (see for example [14])

$$\beta \left( \begin{bmatrix} x_0 \\ y_0 \\ 0 \end{bmatrix}, \begin{bmatrix} F_y \\ -F_x \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix} \right) = \beta \left( \begin{bmatrix} F_y \\ -F_x \\ 0 \end{bmatrix}, \begin{bmatrix} x_1 \\ y_1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix} \right),$$

$\beta$  being the cross-ratio. Therefore we have

$$\frac{(ix_0 - y_0)(-iF_y + F_x)}{(iF_y + F_x)(-ix_0 - y_0)} = \frac{(iF_y + F_x)(-ix_1 - y_1)}{(-iF_y + F_x)(ix_1 - y_1)}$$

and so

$$(ix_0 - y_0)(ix_1 - y_1)(-iF_y + F_x)^2 = (iF_y + F_x)^2(-ix_0 - y_0)(-ix_1 - y_1).$$

Now, according to (i),  $ix_0 - y_0 \neq 0$ ,  $-ix_0 - y_0 \neq 0$ ,  $ix_1 - y_1 \neq 0$ ,  $-ix_1 - y_1 \neq 0$ . Hence  $(-iF_y + F_x)^2 = a(iF_y + F_x)^2$  for some  $a \neq 0$ , which implies that  $d = 1$  and contradicts (ii).

Hence we proved that  $\Sigma_{\mathcal{S}}(\mathcal{C})$  is not reduced to a point. Now the irreducibility of  $\Sigma_{\mathcal{S}}(\mathcal{C})$  comes from the fact that  $\Sigma_{\mathcal{S}}(\mathcal{C}) = \overline{\Phi_{F,S}(\mathcal{C})}$  with  $\Phi_{F,S}$  a rational map and  $\mathcal{C}$  an irreducible curve.

#### APPENDIX A. ABOUT THE FORMULAS OF BROCARD AND LEMOYNE AND OF SALMON AND CAYLEY

Recall that, when  $\mathcal{S} \notin \ell_\infty$ ,  $\Sigma_{\mathcal{S}}(\mathcal{C})$  is the evolute of an homothetic of the pedal of  $\mathcal{C}$  from  $\mathcal{S}$ .

The work of Salmon and Cayley is under ordinary Plücker conditions (no hyper-flex, no singularities other than ordinary cups and ordinary nodes). In [16, p.137], Salmon and Cayley gave the following formula for the class of the evolute :

$$n' = m + n - f - g.$$

Replace now  $m, n, f$  and  $g$  by  $M, N, F$  and  $G$  (respectively) given in [16, p. 154] for the pedal. Doing so, one exactly get (with the same notations) the formula of the class of caustics by reflection given by Brocard and Lemoyne in [1, p. 114].

As explained in introduction, this composition of formulas of Salmon and Cayley is incorrect because of the non-conservation of the Plücker conditions by the pedal transformation. Nevertheless, for completeness sake, let us present the Brocard and Lemoyne formula and compare it with our formula. Brocard and Lemoyne gave the following formula for the class of the caustic by reflection  $\Sigma_{\mathcal{S}}(\mathcal{C})$  when  $\mathcal{S} \notin \ell_\infty$ :

$$\text{class}(\Sigma_{\mathcal{S}}(\mathcal{C})) = d + 2(d^\vee - \hat{f}') - \hat{g} - \hat{f} - \hat{g}' + \hat{q}', \quad (12)$$

for an algebraic curve  $\mathcal{C}$  of degree  $d$ , of class  $d^\vee$ ,  $\hat{g}$  times tangent to  $\ell_\infty$ , passing  $\hat{f}$  times through a cyclic point,  $\hat{f}'$  times tangent to an isotropic line of  $\mathcal{S}$ , passing  $\hat{g}'$  times through  $\mathcal{S}$ ,  $\hat{q}'$  being the coincidence number of contact points when an isotropic line is multiply tangent. In [16],  $\hat{q}'$  is defined as the coincidence number of tangents at points  $\iota_1, \iota_2$  of  $\mathbb{P}^{2^\vee}$  (corresponding to  $(\mathcal{IS})$  and  $(\mathcal{JS})$ ) if these points are multiple points of the image of  $\mathcal{C}$  by the polar reciprocal transformation with center  $\mathcal{S}$ ; i.e.  $\hat{q}'$  represents the number of ordinary flexes of  $\mathcal{C}$ .

When  $\mathcal{S} \notin \ell_\infty$ , let us compare terms appearing in our formula (1) with terms of (12) :

- $\hat{g}$  seems to be equal to  $g$ ;
- it seems that  $\hat{f} = \mu_{\mathcal{I}} + \mu_{\mathcal{J}}$  and so

$$f = \hat{f} + \Omega_{\mathcal{I}}(\mathcal{C}, (\mathcal{IS})) + \Omega_{\mathcal{J}}(\mathcal{C}, (\mathcal{JS}));$$



- it seems that  $\hat{f}' = \sum_{m_1 \in \mathcal{C} \cap (\mathcal{IS})} \Omega_{m_1}(\mathcal{C}, (\mathcal{IS})) + \sum_{m_1 \in \mathcal{C} \cap (\mathcal{JS})} \Omega_{m_1}(\mathcal{C}, (\mathcal{JS}))$  and so

$$f' := \hat{f}' - \Omega_{\mathcal{I}}(\mathcal{C}, (\mathcal{IS})) - \Omega_{\mathcal{J}}(\mathcal{C}, (\mathcal{JS})) - \Omega_{\mathcal{J}}(\mathcal{C}, (\mathcal{JS})) - \Omega_{\mathcal{S}}(\mathcal{C}, (\mathcal{JS}));$$

- it seems that  $\hat{g}' = \mu_{\mathcal{S}}$ , therefore

$$g' := \hat{g}' + \Omega_{\mathcal{S}}(\mathcal{C}, (\mathcal{IS})) + \Omega_{\mathcal{S}}(\mathcal{C}, (\mathcal{JS}));$$

- our definition of  $q'$  appears as an extension of  $\hat{q}'$  (except that we exclude the points  $m_1 \in \{\mathcal{I}, \mathcal{J}, \mathcal{S}\}$ ).

Observe that these terms coincide with the definition of Brocard and Lemoyne if  $(\mathcal{IS})$  and  $(\mathcal{JS})$  are not tangent to  $\mathcal{C}$  at  $\mathcal{S}, \mathcal{I}, \mathcal{J}$ .

**A counterexample to the formula of Brocard and Lemoyne.** Let us consider the non-singular quartic curve  $\mathcal{C} = V(2yz^3 + 2z^2y^2 + 2zy^3 + 2y^4 - 2z^3x + 2zyx^2 + 5y^2x^2 + 3x^4)$  and  $\mathcal{S} = [0 : 0 : 1]$ .

This curve  $\mathcal{C}$  has degree  $d = 4$  and class  $d^\vee = 4 \times 3 = 12$ , is not tangent to  $\ell_\infty$ , is tangent to  $(\mathcal{SI})$  at  $\mathcal{I}$  and nowhere else, is tangent to  $(\mathcal{SJ})$  at  $\mathcal{J}$  and nowhere else; these tangent points are ordinary.  $\mathcal{S}$  is a non singular point of  $\mathcal{C}$ .

Therefore, with our definitions, we have

$$g = 0, \quad f = 2 + 2 = 4, \quad f' = 0, \quad g' = 1 + 1 - 1 = 1, \quad q' = 0.$$

which gives

$$\text{class}(\Sigma_{\mathcal{S}}(\mathcal{C})) = 4 + 2(12 - 0) - 0 - 4 - 1 - 0 = 23,$$

since in this case  $\delta_1(\mathcal{S}, \mathcal{C}) = 1$ .

In comparison, the Brocard and Lemoyne formula would give  $\hat{g} = 0$ ,  $\hat{f} = 1 + 1 = 2$ ,  $\hat{f}' = 1 + 1 = 2$ ,  $\hat{g}' = 1$ ,  $\hat{q}' = 0$  and so  $\text{class}(\Sigma_{\mathcal{S}}(\mathcal{C})) = 4 + 2(12 - 2) - 0 - 2 - 1 - 0 = 21$ .

## APPENDIX B. INTERSECTION NUMBERS OF CURVES AND PRO-BRANCHES

The following result expresses the classical intersection number  $i_{m_1}(\mathcal{C}, \mathcal{C}')$  defined in [13, p. 54] thanks to the use of probranches.

**Proposition 10.** *Let  $m \in \mathbb{P}^2$ . Let  $\mathcal{C} = V(F)$  and  $\mathcal{C}' = V(F')$  be two algebraic plane curves containing  $m$ , with homogeneous polynomials  $F, F' \in \mathbb{C}[X, Y, Z]$ . Let  $M \in GL(\mathbb{C}^3)$  be such that  $\Pi(M(0, 0, 1)) = m$  and such that the tangent cones of  $V(F \circ M)$  and of  $V(F' \circ M)$  do not contain  $X = 0$ .*

*Assume that  $V(F \circ M)$  admits  $b$  branches at  $[0 : 0 : 1]$  and that its  $\beta$ -th branch  $\mathcal{B}_\beta$  has multiplicity  $e_\beta$ . Assume that  $V(F' \circ M)$  admits  $b'$  branches at  $[0 : 0 : 1]$  and that its  $\beta'$ -th branch  $\mathcal{B}'_{\beta'}$  has multiplicity  $e'_{\beta'}$ .*

*Then we have*

$$i_m(\mathcal{C}, \mathcal{C}') = \sum_{\beta=1}^b \sum_{j=0}^{e_\beta-1} \sum_{\beta'=1}^{b'} \sum_{j'=0}^{e'_{\beta'}-1} \text{val}_x[h_\beta(\zeta^j x^{\frac{1}{e_\beta}}) - h'_{\beta'}(\zeta'^{j'} x^{\frac{1}{e'_{\beta'}}})],$$

*with  $y = h_\beta(\zeta^j x^{\frac{1}{e_\beta}}) \in \mathbb{C}\langle x^* \rangle$  an equation of the  $j$ -th probranch of  $\mathcal{B}_\beta$  at  $[0 : 0 : 1]$ ,  $y = h'_{\beta'}(\zeta'^{j'} x^{\frac{1}{e'_{\beta'}}}) \in \mathbb{C}\langle x^* \rangle$  an equation of the  $k'$ -th probranch of  $\mathcal{B}'_{\beta'}$  at  $[0 : 0 : 1]$ , with  $\zeta := e^{\frac{2i\pi}{e_\beta}}$  and  $\zeta' := e^{\frac{2i\pi}{e'_{\beta'}}}$ .*

With the notations of Proposition 10, we get

$$i_m(\mathcal{C}, \mathcal{C}') = \sum_{\beta=1}^b i_{[0:0:1]}(\mathcal{B}_\beta, V(F')), \quad (13)$$

with the usual definition given in [19] of intersection number of a branch with a curve

$$i_{[0:0:1]}(\mathcal{B}_\beta, V(F' \circ M)) = \sum_{j=0}^{e_\beta-1} \text{val}_x(F' \circ M(x, h_{j,\beta}(\zeta^j x^{\frac{1}{e_\beta}}))). \quad (14)$$

*Proof of Proposition 10.* By definition, the intersection number is defined by

$$i_m(\mathcal{C}, \mathcal{C}') = i_{[0:0:1]}(V(F \circ M, F' \circ M)) = \text{length} \left( \left( \frac{\mathbb{C}[X, Y, Z]}{(F \circ M, F' \circ M)} \right)_{(X, Y, Z)} \right)$$

where  $(\frac{\mathbb{C}[X, Y, Z]}{(F \circ M, F' \circ M)})_{(X, Y, Z)}$  is the local ring in the maximal ideal  $(X, Y, Z)$  of  $[0 : 0 : 1]$  [13, p. 53]. According to [11], we have

$$i_m(\mathcal{C}, \mathcal{C}') = \dim_{\mathbb{C}} \left( \left( \frac{\mathbb{C}[X, Y, Z]}{(F \circ M, F' \circ M)} \right)_{(X, Y, Z)} \right)$$

Let  $f, f'$  be defined by  $f(x, y) = F \circ M(x, y, 1)$ ,  $f'(x, y) = F' \circ M(x, y, 1)$ . We get

$$i_m(\mathcal{C}, \mathcal{C}') = \dim_{\mathbb{C}} \left( \left( \frac{\mathbb{C}[x, y]}{(f, f')} \right)_{(x, y)} \right) = \dim_{\mathbb{C}} \frac{\mathbb{C}\langle x, y \rangle}{(f, f')}.$$

Recall that, according to the Weierstrass preparation theorem, there exist two units  $U$  and  $U'$  of  $\mathbb{C}\langle x, y \rangle$  and  $f_1, \dots, f_b, f'_1, \dots, f'_{b'} \in \mathbb{C}\langle x \rangle[y]$  monic irreducible such that

$$f = U \prod_{\beta=1}^b f_\beta \quad \text{and} \quad f' = U' \prod_{\beta'=1}^{b'} f'_{\beta'},$$

$f_\beta = 0$  being an equation of  $\mathcal{B}_\beta$  and  $f'_{\beta'} = 0$  being an equation of  $\mathcal{B}'_{\beta'}$ . According to the Puiseux theorem,  $\mathcal{B}_\beta$  (resp.  $\mathcal{B}'_{\beta'}$ ) admits a parametrization

$$\left\{ \begin{array}{l} x = t^{e_\beta} \\ x = h_\beta(t) \in \mathbb{C}\langle t \rangle \end{array} \right. \quad (\text{resp.} \quad \left\{ \begin{array}{l} x = t^{e'_{\beta'}} \\ x = h'_{\beta'}(t) \in \mathbb{C}\langle t \rangle \end{array} \right).$$

We know that, for every  $\beta \in \{1, \dots, b\}$  and every  $j \in \{0, \dots, e_\beta\}$ ,  $h_\beta(\zeta^j x^{\frac{1}{e_\beta}}) \in \mathbb{C}\langle x^{\frac{1}{e_\beta}} \rangle$  are the  $y$ -roots of  $f_\beta$  (resp.  $h_{\beta'}(\zeta'^{j'} x^{\frac{1}{e'_{\beta'}}}) \in \mathbb{C}\langle x^{\frac{1}{e'_{\beta'}}} \rangle$  are the  $y$ -roots of  $f'_{\beta'}$ ). In particular, we have

$$f_\beta(x, y) = \prod_{j=0}^{e_\beta-1} (y - h_\beta(\zeta^j x^{\frac{1}{e_\beta}})) \quad \text{and} \quad f'_{\beta'}(x, y) = \prod_{j'=0}^{e'_{\beta'}-1} (y - h'_{\beta'}(\zeta'^{j'} x^{\frac{1}{e'_{\beta'}}})).$$

Therefore we have the following sequence of  $\mathbb{C}$ -algebra-isomorphisms:

$$\frac{\mathbb{C}\langle x, y \rangle}{(f, f')} = \frac{\mathbb{C}\langle x, y \rangle}{(\prod_{\beta=1}^b f_\beta(x, y), f'(x, y))} \cong \prod_{\beta=1}^b A_\beta,$$

where  $A_\beta := \frac{\mathbb{C}\langle x, y \rangle}{(f_\beta(x, y), f'(x, y))}$ . Let  $\beta \in \{1, \dots, b\}$ . We observe that we have

$$A_\beta = \prod_{j=0}^{e_\beta-1} \frac{\mathbb{C}\langle x \rangle}{(f'(x, h_\beta(\zeta^j x^{\frac{1}{e_\beta}})))}.$$

On another hand, we have

$$\begin{aligned} D_\beta &:= \frac{\mathbb{C}\langle x^{\frac{1}{e_\beta}}, y \rangle}{(f_\beta(x, y), f'(x, y))} = \frac{\mathbb{C}\langle x^{\frac{1}{e_\beta}}, y \rangle}{(\prod_{j=0}^{e_\beta-1} (y - h_\beta(\zeta^j x^{\frac{1}{e_\beta}})), f'(x, y))} \\ &\cong \prod_{j=0}^{e_\beta-1} \frac{\mathbb{C}\langle x^{\frac{1}{e_\beta}}, y \rangle}{(y - h_\beta(\zeta^j x^{\frac{1}{e_\beta}}), f'(x, y))} \cong \prod_{j=0}^{e_\beta-1} D_{\beta,j} \end{aligned}$$

with

$$D_{\beta,j} := \frac{\mathbb{C}\langle x^{\frac{1}{e_\beta}} \rangle}{(f'(x, h_\beta(\zeta^j x^{\frac{1}{e_\beta}})))}.$$

We consider now the natural extension of rings  $i_\beta : A_{\beta,j} \hookrightarrow D_{\beta,j}$  such that

$$\forall g \in A_\beta, \quad \text{val}_{x^{1/e_\beta}}((i_\beta(g))(x)) = e_\beta \text{val}_x(g(x)).$$

We have

$$D_\beta \cong \prod_{j=0}^{e_\beta-1} \frac{\mathbb{C}\langle x^{\frac{1}{e_\beta}} \rangle}{(x^{v_\beta})},$$

where  $v_\beta$  is the valuation in  $x^{\frac{1}{e_\beta}}$  of  $(f'(x, h_\beta(\zeta^j x^{\frac{1}{e_\beta}})))$ , i.e.

$$v_\beta := \text{val}_t(f'(t^{e_\beta}, h_\beta(\zeta^j t))) = e_\beta \text{val}_x(f'(x, h_\beta(\zeta^j x^{\frac{1}{e_\beta}}))).$$

We get

$$\begin{aligned} i_m(\mathcal{C}, \mathcal{C}') &= \sum_{\beta=1}^b \dim_{\mathbb{C}} A_\beta = \sum_{\beta=1}^b \sum_{j=0}^{e_\beta-1} \frac{1}{e_\beta} \text{val}_t(f'(t^{e_\beta}, h_\beta(\zeta^j t))) \\ &= \sum_{\beta=1}^b \sum_{j=0}^{e_\beta-1} \text{val}_x(f'(x, h_\beta(\zeta^j x^{\frac{1}{e_\beta}}))) = \sum_{\beta=1}^b \sum_{j=0}^{e_\beta-1} \sum_{\beta'=1}^{b'} \text{val}_x(f'_{\beta'}(x, h_\beta(\zeta^j x^{\frac{1}{e_\beta}}))). \end{aligned}$$

Observe now that

$$\text{val}_x(f'_{\beta'}(x, h_\beta(\zeta^j x^{\frac{1}{e_\beta}}))) \in \frac{1}{e_\beta} \mathbb{N}$$

and that

$$f'_{\beta'}(x, h_\beta(\zeta^j x^{\frac{1}{e_\beta}})) \equiv \text{Res}(f'_{\beta'}, f_\beta; y) \equiv \prod_{j'=0}^{e'_{\beta'}-1} (h'_{\beta'}(\zeta'^{j'} x^{\frac{1}{e'_{\beta'}}}) - h_\beta(\zeta^j x^{\frac{1}{e_\beta}})),$$

where  $\text{Res}$  denotes the resultant and where  $\equiv$  means "up to a non zero scalar". Finally, we get

$$i_m(\mathcal{C}, \mathcal{C}') = \sum_{\beta=1}^b \sum_{j=0}^{e_\beta-1} \sum_{\beta'=1}^{b'} \sum_{j'=0}^{e'_{\beta'}-1} \text{val}_x[h'_{\beta'}(\zeta'^{j'} x^{\frac{1}{e'_{\beta'}}}) - h_\beta(\zeta^j x^{\frac{1}{e_\beta}})].$$

□

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## REFERENCES

- [1] H. Brocard, T. Lemoyne. Courbes géométriques remarquables. Courbes spéciales planes et gauches. Tome I. (French) Nouveau tirage Librairie Scientifique et Technique Albert Blanchard, Paris (1967) viii+451 pp.
- [2] J. W. Bruce, P. J. Giblin and C. G. Gibson. *Source genericity of caustics by reflexion in the plane*, Quarterly Journal of Mathematics (Oxford) (1982) Vol. 33 (2) pp. 169–190.
- [3] J. W. Bruce, P. J. Giblin and C. G. Gibson. *On caustics by reflexion*, Topology (1982) Vol. 21 (2) pp. 179–199.
- [4] J. W. Bruce and P. J. Giblin. *Curves and singularities*, Cambridge university press (1984).
- [5] F. Catanese and C. Trifogli. *Focal loci of algebraic varieties. I. Special issue in honor of Robin Hartshorne*. Comm. Algebra 28 (2000), no. 12, pp. 6017–6057.
- [6] M. Chasles. *Détermination, par le principe des correspondances, de la classe de la développée et de la caustique par réflexion d'une courbe géométrique d'ordre m et de classe n* Nouv. Ann. Math. 2 ser. vol. 10 (1871), p. 97–104, extrait C. R. séances A. S. t. LXII.
- [7] J. L. Coolidge. *A treatise on algebraic plane curves*, Dover, Phenix edition (2004).
- [8] G. P. Dandelin. *Notes sur les caustiques par réflexion*, (1822).
- [9] B. Fantechi. *The Evolute of a Plane Algebraic Curve*, (1992) UTM 408, University of Trento.
- [10] G. Fischer. *Plane algebraic curves*, AMS 2001.
- [11] W. Fulton. *Intersection Theory*, 2nd ed., Springer, 1998.
- [12] G. H. Halphen. *Mémoire sur les points singuliers des courbes algébriques planes*. Académie des Sciences t. XXVI (1889) No 2.
- [13] R. Hartshorne. *Algebraic geometry*. Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, (1977).
- [14] A. Josse, F. Pène. *On the degree of caustics by reflection*. Preprint. arXiv:1201.0621.
- [15] L. A. J. Quetelet, *Énoncés de quelques théorèmes nouveaux sur les caustiques*. C. G. Q. (1828) vol 1, p.14, p. 147-149.
- [16] G. Salmon G, *A treatise on higher plane curves: Intended as a sequel to a treatise on conic sections*. Elibron classics (1934).
- [17] C. Trifogli. *Focal Loci of Algebraic Hypersurfaces: a General Theory*, Geom. Dedicata 70 (1998), pp. 1–26.
- [18] E. W. von Tschirnhausen, Acta Erud. nov. 1682.
- [19] C. T. C. Wall. *Singular Points of Plane Curves*. Cambridge University Press. 2004.

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